Solutions to the Boltzmann equation in the Boussinesq regime.

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Abstract We consider a gas in a horizontal slab, in which the top and bottom walls are kept at different temperatures. The system is described by the Boltzmann equation (BE) with Maxwellian boundary conditions specifying the wall temperatures. We study the behavior of the system when the Knudsen number ε is small and the temperature difference between the walls as well as the velocity field is of order ε, while the gravitational force is of order ε^2. We prove that there exists a solution to the BE for $t \in (0, \bar{t})$ which is near a global Maxwellian, and whose moments are close, up to order ε^2 to the density, velocity and temperature obtained from the smooth solution of the Oberbeck-Boussinesq equations assumed to exist for $t \le \bar{t}$.

1. Introduction.

In the study of thermal convection phenomena the following system plays a paradigmatic role: a viscous heat conducting fluid between flat horizontal plates with the lower plate maintained at a temperature greater than the upper one. When the temperature difference between the plates is small, the stationary state is one in which the fluid is at rest with a linear temperature profile. When the temperature difference is made larger, the gravitational buoyancy force acting on the light, higher temperature fluid below, overcomes the effects of viscosity and a new stationary state of thermal Rayleigh-Benard convection sets in. In typical experimental situations the variations of temperature and density are small and the system is described in the Boussinesq approximation under which the Navier-Stokes equations reduce to the Oberbeck-Boussinesq equations (OBE). This approximation, which is

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formulated on a phenomenological basis, gives quantitatively correct predictions in most cases [1].

A justification of this approximation, based on introducing a scaling which leaves the Rayleigh number invariant is given in [2] (see also [3] and [4]). This approach follows the general strategy of taking into account the invariance, under appropriate scaling, of the hydrodynamical equations. Such considerations allow, for example, to derive the incompressible Navier-Stokes equations from the compressible ones, as well as from microscopic and kinetic models.

The main aim of this paper is to derive the OBE starting from the Boltzmann equation (BE), which describes gases on the kinetic level, intermediate between the microscopic and the macroscopic. To go from the kinetic BE to a hydrodynamical one it is necessary to consider situations in which the Knudsen number ε, the ratio between the mean free path and the size of the slab, is very small. It is well known that the inviscid Euler equations correctly describe the behavior of this system for times of order ε^{-1} in the limit $\varepsilon \to 0$ [5]. To obtain the OBE we need to consider longer times, of order ε^{-2}, so as to get the effects of viscosity and thermal conductivity. To make this possible, we have to study the system in the incompressible regime, corresponding to macroscopic velocity fields of order ε [6]. This scaling would appear, at first sight, to require that the force G be scaled as ε^3 to be consistent with the incompressible regime, i.e. to get a finite force term in the equation for the velocity field. However, the case of a conservative force is special from this point of view in that larger forces are permissible. In fact we find that G has to be scaled as ε^2 in order to keep the Rayleigh number finite.

We take the walls to be at fixed temperatures and impose a no-slip boundary condition for the velocity field at the hydrodynamic level. This is modeled at the kinetic level by assuming that each particle colliding with a wall is reflected with a random velocity, distributed according to the equilibrium distribution at the temperature of that wall, i.e. we assume Maxwellian boundary conditions.

Our solution of the BE is given in terms of a truncated expansion in ε whose leading term in the bulk is a global Maxwellian. The term of order ε, denoted by f_1, determines the hydrodynamic quantities which are close, up to order ε^2, to the density, velocity

and temperature solutions of the OBE. Near the boundary, in a thin layer of size ε, the hydrodynamical approximation is not correct and we provide a detailed description of the solution in this region. The initial datum is chosen to match the expansion up to order ε^2, to avoid a treatment of the initial layer which is more or less standard.

The validity of the expansion is established up to a time $\bar{t}$ such that the OBE have a sufficiently regular solution by estimating the remainder. This expansion technique goes back to Hilbert, Chapmann and Enskog; a rigorous proof of the hydrodynamic limit is given for the Euler case in [5] and for the incompressible Navier-Stokes case in [6]. In particular, in the absence of gravity, the results of the present paper extend the ones of [6] to the case of a fluid confined in a domain with walls at different temperatures modeled by Maxwellian boundary conditions. The main technical difficulty for such systems, even in the absence of an external force, is in dealing with the terms coming from the boundary conditions in the estimate for the remainder. The approach proposed in [5], and used also in [6], is based on the estimate of some Sobolev norm of the solutions. But in the presence of the boundaries this cannot be used because the derivatives of the solution may become singular at the boundaries. We avoid the estimates of the derivatives by first looking for L_2 estimates and then improving them to L_∞ estimates. This technique was already used in [7] and [8] which concern essentially one-dimensional problems, i.e. a compressible fluid in a slab with the walls held at fixed temperatures under the action of a force parallel to the walls in the stationary regime. Since we consider here the fully three dimensional time dependent case we need to modify the method to bound the L_∞ norm of the remainder by using a new method based on a result in [9]. This is presented in Section 4 and in the Appendix. A formal expansion including boundary layer corrections was given earlier in [10].

The case with a force perpendicular to the walls presents extra difficulties stemming from the fact that we need good properties of the derivatives with respect to v_z, the vertical component of the velocity, to get the exponential decay of certain boundary layer terms and to control the remainder. Unfortunately, the derivative with respect v_z is singular at the boundaries at $v_z = 0$. To overcome this difficulty we have to decompose the force into a part acting in the bulk only and a part acting only near the boundaries, decreasing

to zero at a distance of order ε. The Milne problem we consider for the boundary layer terms involve these short range forces and can be solved by using the result in [11], where it is proven that the v_z derivative is bounded in L_2 and L_∞ norms locally, away from the boundaries. This is enough to control the terms appearing in the equation for the remainder and in the Milne problems for the higher order corrections.

While the above results are independent of the nature of the solution of the OBE, we are only able to construct stationary solutions to the Boltzmann equation which correspond, at the hydrodynamic level, to the behavior of the purely conducting stationary solution of OBE. This is due to the fact that the methods in this paper are based on the perturbation of a global Maxwellian and therefore require that the temperature difference between the plates be small, corresponding to having a small Rayleigh number. We expect to be able to construct the purely conductive solution of the Boltzmann equation for any Rayleigh number, even when the basic hydrodynamic solution becomes unstable, by perturbing a Maxwellian corresponding to the hydrodynamic solution. This involves many technical difficulties and will be discussed in a forthcoming paper. The hope is to extend these results to the convective solutions which appear for larger values of the Railegh number.

2. Hydrodynamic description.

We consider an incompressible heat-conducting viscous fluid in a horizontal slab $\Lambda = \mathbb{T}_L^2 \times [-1,1]$, where $\mathbb{T}_L^2$ is the two dimensional torus of size L. The acceleration of gravity is the vector $\underline{G} = (0,0,-G)$. We specify the temperature on the boundaries $z = \pm 1$ as:

$$T(1) = T_+, \qquad T(-1) = T_- = T_+ + \delta T. \tag{2.1}$$

The total mass of the fluid is specified to be m with $\bar\rho = m|\Lambda|^{-1}$ the corresponding mass density. We will assume $T_- \geq T_+$. Denote $\theta = T - T_-$ the deviation of the temperature from T_- and set

$$\tilde\theta = \theta - \frac{2}{5}G(1+z).$$

The equations describing the evolution of the velocity and temperature field, u and $\tilde\theta$,

are the Oberbeck-Boussinesq (OBE) equations (see [12], [4]), which we write as

$$\begin{aligned}
&\mathrm{div} u = 0, \\
&\bar\rho(\partial_t u + u\cdot\nabla u) = \eta\Delta u - \nabla\tilde p - \alpha\bar\rho\tilde\theta\underline{G} \\
&\frac{5}{2}\bar\rho(\partial_t\tilde\theta + u\cdot\nabla\tilde\theta) = \kappa\Delta\tilde\theta.
\end{aligned} \tag{2.2}$$

Here η is the kinematic viscosity coefficient, κ the heat conduction coefficient, $\alpha = T_-^{-1}$ the coefficient of thermal expansion,

$$\tilde p = p - \frac{3}{10}\frac{\bar\rho}{RT_-}G^2(1+z)^2$$

and p is the unknown pressure which arises from the incompressibility constraint. The initial conditions are

$$u(\underline{x},0) = u_0(\underline{x}), \qquad \tilde\theta(\underline{x},0) = \theta_0(\underline{x}) - \frac{2}{5}G(1+z) \tag{2.3}$$

for any $\underline{x}\in\Lambda$, with $\mathrm{div} u_0 = 0$. The boundary conditions for this problem are

$$u(x,y,-1,t) = u(x,y,1,t) = 0, \qquad \tilde\theta(x,y,-1,t) = 0, \quad \tilde\theta(x,y,1,t) = -\delta T - \frac{4}{5}G, \tag{2.4}$$

for any $(x,y)\in\mathbb{T}_L^2$ and any positive t.

Moreover, let $r = \rho - \bar\rho$ be the deviation of the density from the homogeneous density $\bar\rho$, let $\tilde r = r + \bar\rho G(1+z)/T_-$. Then $\tilde r$ is determined by the Boussinesq condition

$$\bar\rho\nabla\theta + T_-\nabla\tilde r = 0 \tag{2.5}$$

It can be proved that if u_0 and θ_0 are smooth functions of $\underline{x}$ (i.e. with Sobolev $H_s(\Lambda)$-norm finite for some s sufficiently large), then there is $\bar t$ depending on the initial and boundary data such that the system (2.2), (2.3), (2.4) has an unique solution, at least as smooth as the initial data, for $0 < t \le \bar t$. We do not give the proof of this statement which is rather standard and refer to [4] for details.

The regime under which the OBE are expected to be valid correspond to a low Mach number, a sufficiently weak gravity and a small difference between the temperatures of the

top and bottom walls. To make this percise we introduce a space scale parameter ε and rescale the variables as follows:

$$\underline{x} \to \varepsilon^{-1}\underline{x}, \quad t \to \varepsilon^{-2}t, \quad u \to \varepsilon u, \quad \theta \to \varepsilon\theta, \quad G \to \varepsilon^2 G, \quad \delta T \to \varepsilon\delta T \tag{2.6}$$

This scaling is a natural one in order to derive the OBE because, under it, the Rayleigh number Ra defined as

$$Ra = \left(\frac{GL^3\delta T}{\kappa\nu T_+}\right)^{1/2}.$$

is kept fixed. We refer to [2] for the detailed (formal) derivation of (2.2), (2.3), (2.4), (2.5) from the compressible Navier-Stokes in the limit $\epsilon \to 0$, while in next sections we will provide its rigorous derivation from the Boltzmann equation.

We note that our equations do not coincide exactly with the usual Oberbeck-Boussinesq (OBE) equations as given in [12], [4] because of the term proportional to G in the boundary conditions for $\tilde{\theta}$ and of the quadratic term in G in the definition of $\tilde{p}$. In the usual experimental conditions (see [1]) such terms are much smaller than the others, so one can neglect the effect of the variation of the density due to the gravitational force. If we denote by ρ_s and T_s the solution of the stationary problem

$$\frac{d}{dz}P_s = -G\rho_s, \qquad \Delta T_s = 0,$$

with $P_s = \rho_s T_s$ and boundary conditions (2.1) the approximation corresponds to setting $P_s \sim P(1) \equiv \rho(1)T(1)$. In this way we would recover the usual OBE. We finally remark that the Boussinesq condition (2.5), which is assumed as a "equation of state" in the usual discussions of the Boussinesq approximation (see [12], [3]), in our approach is just a consequence of the scaling limit (2.6).

3. Kinetic description

We consider the BE for a gas between parallel planes. We keep the notations of Sect. 2. To model the hydrodynamic boundary conditions we choose the so called Maxwellian boundary conditions: when a particle hits the walls of the slab ($z = -1$ or $z = 1$) it is

diffusely reflected with a velocity distributed according to a Maxwellian with zero mean velocity and prescribed temperatures T_- and T_+ respectively. In the language of kinetic theory this means that the *accommodation coefficient* equals one. The above prescription implies the impermeability of the walls, namely no particle flux across the boundary is allowed. We introduce as scale parameter ε the Knudsen number. The height of the slab is $2\varepsilon^{-1}$, hence in rescaled variables $z \in [-1, 1]$. We take for simplicity periodic conditions in the x, y direction, and call

$$\Lambda = \{\underline{x} : (x, y) \in \mathbb{T}_L^2, z \in (-1, 1)\}, \qquad \Omega = \{(\underline{x}, v) \,|\, \underline{x} \in \Lambda, v \in \mathbb{R}^3\} \tag{3.1.}$$

The BE rescaled according to (2.6) is

$$\partial_t f^\varepsilon + \varepsilon^{-1} v \cdot \nabla f^\varepsilon + \underline{G} \cdot \nabla_v f^\varepsilon = \varepsilon^{-2} Q(f^\varepsilon, f^\varepsilon), \tag{3.2}$$

with

$$Q(f, f)(\underline{x}, v, t) = \int_{\mathbb{R}^3} dv_* \int_{S_2} d\omega B(\omega, |v - v_*|)\big\{f(\underline{x}, v', t) f(\underline{x}, v'_*, t) - f(\underline{x}, v, t) f(\underline{x}, v_*, t)\big\}$$

where $S_2 = \{\omega \in \mathbb{R}^3 \,|\omega^2 = 1\}$, B is the differential cross section and v',v'_* are the incoming velocities of a collision with outgoing velocities v, v_* and impact parameter ω. We confine ourselves to the collision cross section $B(\omega, V) = |V \cdot \omega|$ corresponding to hard spheres [13].

The initial condition is

$$f^\varepsilon(\underline{x}, v; 0) = f_0^\varepsilon(\underline{x}, v), \quad \underline{x} \in \Lambda, \tag{3.3.}$$

The precise form of f_0^ε will be specified below where it will be seen that it cannot be given arbitrarily if one wants to avoid a detailed analysis of the initial layer. However, we assume the initial datum f_0 non negative and normalized to the total mass which we set to 1.

The boundary conditions are:

$$\begin{aligned} f^\varepsilon(x, y, -1, v; t) &= \alpha_- \overline{M}_-(v), \quad v_z > 0, \;\; t > 0, \\ f^e(x, y, 1, v; t) &= \alpha_+ \overline{M}_+(v), \quad v_z < 0, \;\; t > 0, \end{aligned} \tag{3.4}$$

with

$$\overline{M}_{\pm}(v) = \frac{1}{2\pi T_{\pm}^2} \mathrm{e}^{-v^2/2T_{\pm}}, \tag{3.5}$$

normalized so that $\int_{v_y \lessgtr 0} |v_y| \overline{M}_{\pm}(v) dv = 1$. The temperature T_+ is assumed to satisfy

$$T_+ = T_-(1 - 2\varepsilon\lambda)$$

with λ independent of ε, according to the scaling (2.6).

The quantities $\alpha_{\pm}$ must be chosen in such a way that the impermeability condition of the walls is assured, i.e.

$$\langle v_z f^\varepsilon \rangle \equiv \int_{\mathbb{R}^3} v_z f^\varepsilon dv = 0 \qquad \text{for } z = \pm 1, \tag{3.6}$$

where we have introduced the notation $\langle f \rangle = \int_{\mathbb{R}^3} f(v) dv$. Condition (3.6) and the normalization of $\overline{M}_{\pm}$ imply:

$$\alpha_{\pm} = \pm \int_{v_z \gtrless 0} v_z f^\varepsilon(x, y, \pm 1, v; t) dv \tag{3.7}$$

Namely, $\alpha_{\pm}$ represent the outgoing (from the fluid) fluxes of mass in the direction z. The impermeability condition implies that the normalization of the solution to (3.2) stays constant and therefore we will look for solutions to (3.2) which are normalized to 1 as the initial datum.

The macroscopic behavior should be recovered in the limit ε going to zero. More precisely we expect that for ε small the behavior of the solution (3.2) is very close to the hydrodynamical one, in the sense that it can be described by a local Maxwellian with parameters which differ from constants by terms of order ε, and that these terms are solution of the OBE. At higher order in ε there will both be kinetic and boundary layer corrections. Therefore we look for a solution of the form

$$f^\varepsilon = M + \varepsilon f_1 + \sum_{n=2}^{7} \varepsilon^n f_n + \varepsilon^4 R \tag{3.8}$$

where M is the global Maxwellian

$$M(\bar{\rho}, 0, T_-; v) = \frac{\bar{\rho}}{(2\pi T_-)^{3/2}} \mathrm{e}^{-v^2/2T_-}.$$

If we put (3.8) in the BE (3.2) we see immediately that f_1 has to satisfy

$$\mathcal{L} f_1 := 2Q(M, f_1) = 0, \tag{3.9}$$

where $\mathcal{L}$ is the linearized Boltzmann operator. (3.9) implies that f_1 has to be in Null $\mathcal{L}$, which means that it is a combination of the collision invariants $M\chi_i$ with $\chi_i(v) = 1, v_i, (v^2 - 3T_-)/2$, for $i = 0$, $i = 1, 2, 3$ and $i = 4$ respectively, suitably normalized to form an orthonormal set, in $L_2(M(v)^{-1}dv)$. Hence we have

$$f_1 = M \sum_{i=0}^{4} \chi_i t_i(t, \underline{x}) \equiv M\Big(\frac{r}{\bar\rho} + \frac{u \cdot v}{T_-} + \theta \frac{|v|^2 - 3T_-}{2T_-^2}\Big). \tag{3.10}$$

The functions $t_i(t, \underline{x})$ and/or r, u, θ will satisfy equations to be determined. To write the conditions for f_n we decompose them into two parts B_n and $b_n^{\pm}$, representing the bulk and boundary layer corrections. The latter are significantly different from 0 only near the boundary. The B_n have to satisfy for $n = 2, \cdots, 7$

$$\partial_t B_{n-2} + v \cdot \nabla B_{n-1} + \underline{G} \cdot \nabla_v B_{n-2} = 2Q(M, B_n) + \sum_{i+j=n} Q(B_i, B_j) \tag{3.11}$$

where $B_0 \equiv M$ and $B_1 \equiv f_1$.

We note that the condition $f_1 = B_1$ means that there is no boundary layer correction to the first order in ε. To make this compatible with (3.4) we need to assume that $u(x, y, -1, t) = u(x, y, 1, t) = 0$, $\theta(x, y, -1, t) = 0$, $\theta(x, y, 1, t) = 2\lambda T_-$ for any $(x, y) \in \mathbb{T}_L$ and any $t > 0$. We remark that $M + \varepsilon f_1$, when evaluated for $z = 1$ is not proportional to the Maxwellian $\bar{M}_+$, even with previous assumptions, but differs from it for terms of order ε^2 which will appear in the corrections of higer order.

To construct the boundary layer terms we decompose the constant gravity force $\underline{G} = (0, 0, -G)$ into three parts: a bulk part $\underline{G}_0$ and two boundary parts $\underline{G}^{\pm}$ which are different from zero in the bulk and near the walls respectively. Their definition is

$$\underline{G} = \underline{G}^+ + \underline{G}^0 + \underline{G}^-$$

with $\underline{G}_0$ and $\underline{G}^{\pm}$ smooth functions such that

$$\underline{G}^+(z) = \begin{cases} \underline{G}, & 1 - \delta\, \varepsilon \le z \le 1 \\ 0, & -1 \le z \le 1 - 2\delta\, \varepsilon \end{cases}, \qquad \underline{G} = \begin{cases} \underline{G}, & -1 \le z \le -1 + \delta\, \varepsilon \\ 0, & -1 + 2\delta\, \varepsilon \le z \le 1 \end{cases} \tag{3.12}$$

$$\underline{G}(z) = \begin{cases} \underline{G}, & -1 + 2\delta\,\varepsilon \le z \le 1 - 2\delta\,\varepsilon \\ 0, & |z| \ge 1 - \delta\,\varepsilon \end{cases}. \tag{3.13}$$

Moreover we have to scale back to microscopic coordinates around $z = \pm 1$. Setting $z^{\pm} = \varepsilon^{-1}(1 \mp z)$ so that $z^{\pm} \in [0, 2\varepsilon^{-1}]$ we have that $\underline{G}^{\pm}(z^{\pm})$ is zero for $z^{\pm} \in [2\delta, 2\varepsilon^{-1}]$. The boundary layer corrections relative to the wall $z = \pm 1$, $b_n^{\pm}$, are chosen to satisfy, for $n = 2 \dots 5$, the equations

$$\begin{aligned} v_z \frac{\partial b_n^{\pm}}{\partial z^{\pm}} \mp \varepsilon^2 G^{\pm} \frac{\partial}{\partial_{v_z}} b_n^{\pm} &= \partial_t b_{n-2}^{\pm} + \hat{v} \cdot \hat{\nabla} b_{n-1}^{\pm} + \mathcal{L}^{\pm} b_n^{\pm} + 2Q(\Delta M, b_{n-1}^{-})\chi^{\pm} \\ &\mp (G^0 + G^{\mp}) \frac{\partial}{\partial v_z} b_{n-2}^{\pm} + \sum_{\substack{i,j \ge 1 \\ i+j=n}} \Big[2Q(B_i, b_j^{\pm}) + Q(b_i^{\pm}, b_j^{\pm}) + Q(b_i^{\mp}, b_j^{\mp}) \Big], \end{aligned} \tag{3.14}$$

where we have put

$$b_0^{\pm} = b_1^{\pm} = 0, \qquad \hat{v} = (v_x, v_y), \qquad \hat{\nabla} = (\partial_x, \partial_y), \qquad \chi^{+} = 1, \chi^{-} = 0$$

$$\mathcal{L}^{\pm} = 2Q(M^{\pm}, \cdot), \qquad \Delta M = \varepsilon^{-1}[M - M_+], \qquad M_+ = M(\rho_+, 0, T_+; v),$$

and $\rho_+ = \bar{\rho} + \varepsilon r(1)$.

Finally the equation for the remainder is

$$\partial_t R + \frac{1}{\varepsilon} v \cdot \nabla R + \underline{G} \cdot \nabla_v R = \frac{1}{\varepsilon^2} \mathcal{L} R + \frac{1}{\varepsilon} \mathcal{L}^{(1)} R + \mathcal{L}^{(2)} R + \varepsilon^2 Q(R, R) + \varepsilon^2 A \tag{3.15}$$

with

$$\mathcal{L}^{(1)} R = 2Q(f_1, R), \quad \mathcal{L}^{(2)} R = 2Q(\sum_{n=2}^{7} \varepsilon^{n-2} f_n, R) \tag{3.16}$$

and A given by

$$\begin{aligned} A = &- \partial_t (f_6 + \varepsilon f_7) - v \cdot \nabla B_7 - \hat{v} \cdot \hat{\nabla} b_7^{+} - \hat{v} \cdot \hat{\nabla} b_7^{-} - \underline{G} \cdot \nabla_v (B_6 + \varepsilon B_7) \\ &- (\underline{G}^0 + \underline{G}^{-}) \cdot \nabla_v [(b_6^{+} + \varepsilon b_7^{+})] - (\underline{G}^0 + \underline{G}^{+}) \cdot \nabla_v [(b_6^{-} + \varepsilon b_7^{-})] + \\ &2Q(\Delta M, b^{-}) + \sum_{\substack{k,m \ge 1 \\ k+m \ge 8}} \varepsilon^{k+m-8} Q(f_k, f_m) \end{aligned} \tag{3.17}$$

The boundary conditions for these equations have to be chosen in such a way as to satisfy (3.4)–(3.6) for f^{ε}. Since we are interested in the case $T_+ = T_-(1 - 2\varepsilon\lambda)$ it is

easy to satisfy (3.4) up to the first order in ε, because M is already a Maxwellian whose temperature and velocity field are chosen to fit with $\overline{M}_-$, while $M + \varepsilon f_1$ is close to be proportional to $\overline{M}_+$ at $z = 1$, up to terms of order ε^2.

From the second order on we have to use boundary layer terms to fit boundary conditions. In fact, as we will see later, the B_n, for $n \geq 2$, do not reduce to $\alpha_n^{\pm}\overline{M}_{\pm}$. The idea is to introduce at one of the boundaries, say $z = 1$, the correction b_2^+ so that $B_2 + b_2^+$ is proportional to $\overline{M}_+$ for $v_z < 0$. On the other hand, the same has to be done at $z = -1$ and f_2 is modified by b_2^- also. This changes again f_2 at $z = 1$ by non Maxwellian terms. However, since b_2^- decays exponentially fast, the modification is exponentially small in ε^{-1}. Therefore we impose on the f_n the following boundary conditions:

$$\begin{aligned} f_n(\underline{x}, v; t) &= \alpha_n^- \overline{M}_-(v) + \gamma_{n,\varepsilon}^-(v), \quad z = -1, \quad v_z > 0, \ \ t > 0 \\ f_n(\underline{x}, v; t) &= \alpha_n^+ \overline{M}_+(v) + \gamma_{n,\varepsilon}^+(v), \quad z = 1, \quad v_z < 0, \ \ t > 0 \end{aligned} \tag{3.18}$$

with the functions $\gamma_{n,\varepsilon}^{\pm}(v)$ exponentially small in ε^{-1} and such that $\langle \gamma_{n,\varepsilon}^{\pm} v_z \rangle = 0$, to be specified later. Moreover

$$\alpha_n^{\pm} = \pm \int_{v_z \gtrless 0} v_z f_n(x, y, \pm 1, v; t) dv \tag{3.19}$$

Finally, to fulfil (3.4) we impose the following conditions on R:

$$R(\underline{x}, v; t) = \alpha_R^- \overline{M}_-(v) - \sum_{n=2}^{7} \varepsilon^{n-3} \gamma_{n,\varepsilon}^-, \quad z = -1, \quad v_z > 0, \ \ t > 0 \tag{3.20}$$

$$R(\underline{x}, v; t) = \alpha_R^+ \overline{M}_+(v) - \sum_{n=2}^{7} \varepsilon^{n-3} \gamma_{n,\varepsilon}^+, \quad z = 1, \quad v_z < 0, \ \ t > 0 \tag{3.21}$$

The initial conditions for $R(\underline{x}, v; 0)$ are chosen to be $R(\underline{x}, v; 0) = 0, \ \ z \neq \pm 1$ for simplicity. The initial values for the f_n's are partly determined by the procedure below, so that only their hydrodynamical part can be assigned arbitrarily. To remove such restrictions one would have to include an analysis of the initial layer, which we skip to make the presentation simpler. Finally we impose the conditions

$$\int_\Omega d\underline{x} dv f_n = 0 = \int_\Omega d\underline{x} dv R \tag{3.22}$$

to ensure the normalization of the solution. Note that this condition on R is satisfied because it is true at time $t = 0$.

Outline of Solution.

The equations for the f_n are coupled in a complicated way and have to be solved in the proper sequence, which we now outline. The hydrodynamical part of the bulk terms is determined by the solvability conditions for (3.11), that we get by multiplying (3.11) by $\chi_i, i = 0 \ldots 4$, integrating over velocity and using the fact that $\langle Q(f,g)\chi_i\rangle = 0$. The solvability condition for (3.11) with $n = 2$ is

$$\langle \chi_i [v \cdot \nabla f_1 + \underline{G} \cdot \nabla_v M]\rangle = 0, \quad i = 0, \cdots, 4 \tag{3.23}$$

because the Maxwellian M does not depend on x and t. This is equivalent to

$$\mathrm{div} u = 0, \qquad \bar\rho \nabla\theta + T_- \nabla r = \bar\rho G. \tag{3.24}$$

The first one is the usual incompressibility condition while the second one becomes the Boussinesq condition (2.5), when one defines $\tilde r = r + \bar\rho(G/T_-)(1+z)$. Once (3.24) are satisfied, we can deduce from (3.11) with $n = 2$ the following expression for B_2, where $\mathcal{L}^{-1}$ denotes the inverse of the restriction of $\mathcal{L}$ to the orthogonal of its null space

$$B_2 = \mathcal{L}^{-1}\Big[v \cdot \nabla f_1 + \underline{G} \cdot \nabla_v M - Q(f_1, f_1)\Big] + M \sum_{i=0}^{4} \chi_i \; t_i^{(2)}(t, \underline{x}) \tag{3.25}$$

The solvability condition for (3.11) with $n = 3$ is

$$\langle \chi_i [\partial_t f_1 + \underline{G} \cdot \nabla_v f_1 + v \cdot \nabla B_2]\rangle = 0, \quad i = 0, \cdots, 4 \tag{3.26}$$

and this produces the equations for u and θ. Let us fix $i = 1, 2, 3$ in (3.26). Then the first term gives the time derivative of $\bar\rho u$. The second one reduces to $-\underline{G}\hat r$ after integrating by parts. Finally we write

$$\langle v \otimes v B_2\rangle = \langle [v \otimes v - \frac{v^2}{3}\mathbb{I}]B_2\rangle + \langle \frac{v^2}{3}\mathbb{I} B_2\rangle$$

The first term, as is well known, gives rise to the dissipative and transport terms in the second of (2.2), while the second one is the second order correction to the pressure P_2. The result is

$$\bar{\rho}(\partial_t u + u \cdot \nabla u) = \nu \Delta u - \nabla P_2 + Gr.$$

Using the Boussinesq condition, the definitions of r and θ and the relation between P_2 and p of Section 2, we find $(2.2)_2$ as in Section 2, with η given by

$$\eta = \langle (v \otimes v - \frac{v^2}{3}\mathbb{I})\mathcal{L}^{-1}[M(v \otimes v - \frac{v^2}{3}\mathbb{I})]\rangle.$$

To get the equation for the temperature, it is convenient to replace χ_4 in (3.26) by $\hat{\chi}_4 = \frac{1}{2}[v^2 - 5T_-)]$. A simple computation, using the Boussinesq condition, yields:

$$\langle \frac{1}{2}[v^2 - 5T_-)] f_1 \rangle = \frac{5}{2}\bar{\rho}\big[\theta - (\frac{2}{5}G)\big] + \text{const.},$$

$$\langle \frac{1}{2}[v^2 - 5T_-)\underline{G} \cdot \nabla_v f_1 \rangle = -\bar{\rho} u_z G,$$

$$\langle v \frac{1}{2}[v^2 - 5T_-)] B_2 \rangle = -\kappa \nabla \theta + \frac{5}{2}\bar{\rho} u \theta (1 - z)).$$

Collecting the above results we get (2.2) with κ given by

$$\kappa = \langle v \frac{1}{2}(v^2 - 5T_-)\mathcal{L}^{-1}[M v \frac{1}{2}(v^2 - 5T_-)]\rangle. \tag{3.27}$$

Finally, equation (3.26) with $i = 0$ gives

$$\partial_t r = \text{div}\ \underline{t}^{(2)}, \qquad \underline{t}^{(2)} = (t_1^{(2)}, t_2^{(2)}, t_3^{(2)}). \tag{3.28}$$

Summarizing our results so far: we have shown that, assuming u, p, θ satisfy the OBE (2.2), (2.3), (2.4), (2.5) up to a time $\bar{t}$, the coefficients t_i entering in the definition of f_1 are determined. Therefore, once initial and boundary conditions for the OBE are specified, f_1 is completely determined as a function of $(t, \underline{x}, v)$.

On the other hand the hydrodynamic part of B_2 is not yet determined, but, by (3.28), div $\underline{t}^{(2)}$ is determined in terms of r. Moreover, a combination of $t_0^{(2)}$ and $t_4^{(2)}$ contributes to the pressure p which is determined by the OBE, so that these parameters are not independent.

The non-hydrodynamic part of B_2 depends on the derivatives of r, u, θ which are in general different from zero on the boundaries. Therefore B_2 violates the boundary conditions and we need to introduce $b_2^{\pm}$ to adjust the boundary conditions. We choose b_2^- by solving, for any $t > 0$, the Milne problem

$$v_z \frac{\partial}{\partial z} h - \varepsilon^2 G^- \frac{\partial}{\partial v_z} h = \mathcal{L}^- h, \quad \langle v_z h \rangle = 0$$

with boundary condition (at $z^- = 0$) prescribing the incoming flux as the opposite of the non hydrodynamic part of B_2 at $z = -1$. The results in [11] tell us that the solution approaches, as $z^- \to \infty$ a function q_2^- in Null $\mathcal{L}^-$. Thus we set $b_2^- = h - q_2^-$, which will go to zero at infinity exponentially in z^- and will be the boundary layer correction we are looking for.

In conclusion, we have

$$f_2(\underline{x}, v; t) = M \sum_{i=0}^{4} t_i^{(2)}(x, y, -1; t)\chi_i(v) + b_2^+(x, y, 2\varepsilon^{-1}; t) - q_2^-, \quad z = -1, \quad v_z > 0, \quad t > 0$$

Since the coefficients $t_i^{(2)}$ can be chosen arbitrarily on the boundaries we use them to compensate q_2^-. To satisfy the impermeability conditions we have to choose $t_3^{(2)} = 0$ on the boundaries. The coefficients of the hydrodynamic part of B_2 will, for $i \neq 0$, be determined by the compatibility condition for $n = 4$. These are time-dependent non-homogeneous Stokes equations (linear second order differential equations) in a slab, together with the b. c. $t_i^{(2)} = q_{2i}^-$, $i = 1, 2, 4$. Then $t_0^{(2)}$ is found up to a constant that is chosen so that the total mass associated to f_2 vanishes. Finally we get

$$f_2(x, y, \pm 1, v_z \gtrless 0; t) = \alpha_2^{\pm} M_{\pm} + \gamma_{2,\varepsilon}^{\pm}, \quad \alpha_2^{\pm} = t_0^{(2)}(x, y, \pm 1) - q_2^{(\pm)}(0)$$

Iterating this procedure it is possible to find all f_n. To prove that the terms in the expansion have the right properties we use the results in [11] for the solutions of the Milne problem with a force, that we state below

Let $F(z) = -\nabla V(z)$ be a force vanishing at infinity such that $V(x)$ and its derivative are bounded. Define

$$\tilde{M} = e^{-V(z)} M, \qquad \tilde{L} f = e^{-V(z)} \frac{1}{\sqrt{M}} 2Q(\sqrt{M} f, M).$$

Consider the following Milne problem:

$$v_z \frac{\partial f}{\partial z} + F(z)\frac{\partial f}{\partial v_z} = \tilde{L}f + s(z,v), \qquad 0 < z < +\infty \tag{3.29}$$

$$f(0,v) = h(v), \qquad v_z > 0 \tag{3.30}$$

$$\lim_{z\to+\infty} f(z,v_z) = l < +\infty \tag{3.31}$$

$$\int dv v_z \sqrt{M} f = 0 \tag{3.32}$$

$$\int dv \sqrt{M} s = 0 \tag{3.33}$$

In [11] it is proven the following

Theorem 3.1

1) Suppose that for $r > 3$ and some $\sigma' > 0$ there are finite constants c_1 and c_2 such that

$$\sup_{v\in\mathbb{R}^3} (1+|v|)^r |h(v)| < c_1$$

$$\sup_{z\in\mathbb{R}^+} e^{\sigma' z} \sup_{v\in\mathbb{R}^3} (1+|v|)^r |s(z,v)| < c_2 \tag{3.34}$$

Then there is a unique solution $f \in L_\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ to the Milne problem (3.29)–(3.32). Moreover there exist constants c and c' such that f verifies the conditions:

$$f_\infty \in Null\tilde{L} \tag{3.35}$$

$$\sup_{z\in\mathbb{R}^+} e^{\sigma z} \sup_{v\in\mathbb{R}^3} (1+|v|)^r |(f(z,v) - f_\infty(v))| < c \tag{3.36}$$

for any $\sigma < c'$.

2) Suppose that for fixed $r > 3$, $\ell \geq 1$ and some $\beta > 0$

$$\sup_{v\in\mathbb{R}^3} (1+|v|)^r \Big|\frac{\partial^\ell h}{\partial v_i^\ell}\Big| + \sup_{z\in\mathbb{R}^+} e^{\beta z} \sup_{v\in\mathbb{R}^3} (1+|v|)^r \Big|\frac{\partial^\ell s}{\partial v_i^\ell}\Big| < c_\ell \tag{3.37}$$

for some constant c_ℓ and $i \neq 3$. Then there are finite constants c and c'_ℓ such that

$$\sup_{z\in\mathbb{R}^+} e^{\beta z} \sup_{v\in\mathbb{R}^3} (1+|v|)^r \Big|[\frac{\partial^\ell f}{\partial v_i^\ell} - \frac{\partial^\ell f_\infty}{\partial v_i^\ell}]\Big| < c'_\ell \tag{3.38}$$

for any $\sigma < c$.

3) If $\gamma := \sup_{z\in(0,+\infty)}[|F'|+|F|]$ exists and is finite then for any $\delta > 0$ and for γ sufficiently small there exists a finite constant C_δ such that

$$\sup_{v\in R^d}\sup_{z\in(\delta,+\infty)}(1+|v|)^r\Big|\frac{\partial}{\partial z}f\Big| \le C_\delta, \qquad \sup_{v\in R^d}\sup_{z\in[\delta,+\infty)}(1+|v|)^r\Big|[\frac{\partial f}{\partial v_z}-\frac{\partial f_\infty}{\partial v_z}]\Big| \le C_\delta \quad (3.39)$$

4. Results in the time-dependent case.

The main properties of the f_n's are summarized in Theorem 4.1 below.

Theorem 4.1

Suppose that there is $\bar{t} > 0$ such that $p(t)$, $\theta(t)$ and $u(t)$ are smooth solutions of OBE, with $||\nabla u(t)||_{H_s} + ||\nabla\theta(t)||_{H_s} \le q$ for sufficiently large s and $0 < t \le \bar{t}$. Then it is possible to determine functions f_n, $n = 2,\dots,7$ satisfying, for $0 < t \le \bar{t}$, equation (3.11) and the conditions

$$f_n(\underline{x}, v; 0) = f_n^0, \qquad f_n(x, y, \pm 1, v_x, v_y, v_z \gtrless 0; t) = \alpha_n^\pm M_\pm, \quad t > 0, \tag{4.1}$$

$$\langle A\rangle = 0 \tag{4.2}$$

$$\int_{\mathbb{R}^3\times\mathbb{T}^2\times[-1,1]} dvdxdydzf_n = 0 \tag{4.3}$$

Moreover, for any $\ell \ge 3$ there is a constant c such that:

$$\sup_{0\le t\le\bar{t}} |f_n|_{\ell,h} < cq \tag{4.4}$$

$$\sup_{0\le t\le\bar{t}} |A|_{\ell,h} < cq, \tag{4.5}$$

for $h \le 1/(4T_-)$. Here

$$|f|_{\ell,h} = \sup_{\underline{x}\in\mathbb{R}^2\times[-1,1]}\sup_{v\in\mathbb{R}^3}(1+|v|)^\ell \exp[hv^2]|f(\underline{x}, v; t)| \tag{4.6}$$

Proof

The proof is achieved by showing that every step of the procedure described in the previous section is correct, namely that the conditions on the source and on the boundary conditions for the Milne problems that we have to solve at each step are satisfied. Moreover we need to check the solvability conditions for the Stokes equations.

Step 1

The first step is finding the boundary layer term b_2^+ solving for any $(x,y) \in \mathbb{T}^2$ the Milne problem for $g_2 = b_2^+/\sqrt{\tilde{M}_+}$:

$$v_z \frac{\partial}{\partial z^+} g_2 + \varepsilon^2 G^+ \frac{\partial g_2}{\partial v_z} = \mathcal{L}^+ g_2, \tag{4.7}$$

$$\sqrt{\tilde{M}_+} g_2(x,y,0,v;t) = \bar{B}_2(x,y,1,v;t) - q_2^+(x,y,v;t), \quad v_z > 0, t > 0, \qquad \langle v_z g \rangle = 0, \tag{4.8}$$

where $\tilde{M}_+ = \exp[-V^+(z^+)] M_+$, $-\partial_{z^+} V^+ = \varepsilon^2 G^+$, $\bar{B}_2$ is the non-hydrodynamical part of B_2 given by $\bar{B}_2 = \mathcal{L}^{-1}\Big[v \cdot \nabla f_1 + \underline{G} \cdot \nabla_v M - Q(f_1, f_1)\Big]$. Finally $q_2^+(v;t)$ is the limit at infinity of the solution $\tilde{b}_2^+$ of the same Milne problem with boundary condition $\bar{B}_2$, as explained in the previous section.

The force G^+ has been chosen smooth and vanishing as z^+ goes to $+\infty$ in such a way as to satisfy the assumptions on the force in Theorem 3.1. Furthermore, for ε small the force term (and its derivative) in (4.7) is small. The boundary conditions verify (3.34) by the property of $\mathcal{L}^{-1}$ (see [13]). Hence, by Theorem 3.1, $\tilde{M}_+^{-1/2} b_2^+$ satisfy (3.35)–(3.39).

In the same way we construct b_2^- imposing the boundary condition in $z^- = 0$ given by $\bar{B}_2(x,y,-1,v;t)$.

Step 2

As explained above the coefficients $t_i^{(2)}, i = 1,2,4$ of the hydrodynamical part of B_2 are determined by the compatibility condition for $n = 4$

$$\big\langle \chi_i \big[\partial_t B_2 + v \cdot \nabla B_3 + G \cdot \nabla_v B_2\big]\big\rangle$$

where

$$B_3 = \mathcal{L}^{-1}\Big[\partial_t f_1 + v \cdot \nabla B_2 + \underline{G} \cdot \nabla_v f_1 - 2Q(f_1, B_2)\Big] + M \sum_{i=0}^{4} \chi_i \, t_i^{(3)}(\underline{x}, t) \tag{4.9}$$

Proceeding as in the determination of the Boussinesq equation, we find now a set of three linear time-dependent non-homogeneous Stokes equations for $t_i^{(2)}$. The non-homogeneous terms depend on the third order spatial derivatives of f_1. We note that the non linear terms in the hydrodynamic equations come from the quadratic term $Q(f_1, f_1)$ in (3.25), while in (4.9) appears a term linear in B_2. General theorems for the Stokes equation assures the existence of a solution for the chosen boundary and initial conditions.

Step 3

Once B_2 is completely determined, (4.9) gives the non-hydrodynamical part of B_3, $\bar{B}_3$. As before, we introduce the terms $b_3^{\pm}$ to compensate $\bar{B}_3$ on the boundaries $z = \pm 1$. The term b_3^+ is found as a solution of the Milne problem for g_3 : $\sqrt{\tilde{M}_+} g_3 = b_3^+$

$$v_z \frac{\partial g_3}{\partial z^+} + \varepsilon^2 G^+ \partial_{v_z} g_3 = \mathcal{L}^+ g_3 + s(x, y, z^+, v; t) \tag{4.10}$$

with source

$$\sqrt{\tilde{M}_+} s(x, y, z^+, v; t) = \hat{v} \cdot \hat{\nabla} b_2^+ + 2Q(\Delta M, b_2^+) + 2Q(f_1, b_2^+) \tag{4.11}$$

and with boundary condition

$$\sqrt{\tilde{M}_+} g_3(x, y, 0, v; t) = \bar{B}_3(x, y, 1, v; t) - q_3^+(v; t) \quad v_z > 0, t > 0.$$

We have to check that the source satisfies the conditions of Theorem 3.1.

The condition (3.33) is true due to the properties of Q and to (3.32) for b_2^+. The terms of the form $Q(f, g)$ are bounded by means of the Grad estimates

$$|M^{-\frac{1}{2}} Q(f, g)|_{r-1} \le C ||M^{-\frac{1}{2}} f|_r||M^{-\frac{1}{2}} g|_r \tag{4.12}$$

so that

$$|\tilde{M}_-^{-\frac{1}{2}} Q(f_1, b_2^+)|_{r-1} \le C|\tilde{M}_-^{-\frac{1}{2}} f_1|_r |\tilde{M}_-^{-\frac{1}{2}} b_2^+|_r$$

The second term in (4.11) is bounded in the same way

$$|\tilde{M}_-^{-\frac{1}{2}} Q(\Delta M, b_2^+)|_{r-1} \le \left| \varepsilon^{-1} \frac{M - M_-}{\sqrt{\tilde{M}_-}} \right|_r |\tilde{M}_-^{-\frac{1}{2}} b_2^-|_r$$

Since $T_- = T_+(1+2\varepsilon\lambda)$ we have that $|\varepsilon^{-1}\frac{M-M_+}{\sqrt{\tilde{M}_-}}|_{r-1} \le C|(\frac{v^2}{T_-}+1)\sqrt{M}|_{r-1}$.

The functions f_1 and b_2^+ have bounded norm, hence the third term in (4.11) is bounded. To conclude the step we need to bound the first term. It is bounded by using the properties for the derivatives with respect to x and y of b_2^+ assured by Theorem 3.1.

Step 4

B_3 is constructed as in Step 2. Instead the Milne problem for b_4^+ has to be discussed since in the source appear also derivatives of b_2^+ with respect t and v_z. We have for $\sqrt{\tilde{M}_+}g_4 = b_4^+$ that

$$v_z\frac{\partial g_4}{\partial z^+} + \varepsilon^2 G^+\frac{\partial}{\partial v_z}g_4 = \mathcal{L}^+ g_4 + s_4,$$

$$\begin{aligned}\sqrt{\tilde{M}_+}s_4(x,y,z^+,v;t) = \partial_t b_2^+ - (G^0+G^-)\frac{\partial}{\partial v_z}b_2^+ + \hat{v}\cdot\hat{\nabla}b_3^+ + 2Q(\Delta M, b_3^+) + \\ \sum_{\substack{i,j\ge 1\\ i+j=4}}\Big[2Q(B_i,b_j^\pm) + Q(b_i^\pm,b_j^\pm) + Q(b_i^\mp,b_j^\mp)\Big].\end{aligned} \tag{4.13}$$

The first term in r.h.s. of (4.13) is bounded by observing that the time derivative of b_2^- is a solution of the Milne problem we get by differentiating (4.7) with respect to time with boundary condition $\partial_t b_2^-(0,v;t)$, $v_z>0$, $t>0$.

The second term in r.h.s. of (4.13) is zero by construction for $0\le z^-\le\delta$ so that

$$|\tilde{M}_-^{-1/2}(G^0+G^-)\nabla_{v_z}b_2^+|_{r-1} \le C\Big\{1+\sup_{v\in R^d}\sup_{x\in(\delta,+\infty)}(1+|v|)^r\Big|[\frac{\partial g_2}{\partial v_z}-\frac{\partial g_{2\infty}}{\partial v_z}]\Big|\Big\}$$

and it is bounded because of (3.39), Theorem 3.1, that applies as shown in Step 1.

Finally the condition (3.33) is satisfied since the velocity flux of the derivatives of b_2^- with respect to time and velocity is zero.

The terms f_n with higher n are constructed and bounded in the same way. As a consequence, it is easy to see by using the preceding arguments that the term A in (3.15) satisfies the bound (4.5) and (4.2). This concludes the proof of Theorem 4.1.

To complete the construction of a solution to the BE, we have to show that the remainder is bounded in norm $|\cdot|_{\ell,h}$ defined in (4.6). The remainder has to satisfy (3.15) and the

conditions (3.20) and (3.21) with $\langle \gamma^{\pm}_{n,\varepsilon} v_z \rangle = 0$. Moreover R has to satisfy

$$R(\underline{x}, v; 0) = 0., \qquad \langle v_z R \rangle = 0 \ \text{ in } z = \pm 1 \tag{4.14}$$

which implies $\alpha^{\pm}_R = \pm \int_{v_z \gtrless 0} v_z R(\pm 1, v; t) dv$. To construct the solution of (3.15),(3.20), (3.21) and (4.14) we first deal with the following linear initial boundary value problem: given D on $\mathbb{T}^2 \times [-1, 1] \times \mathbb{R}^3$, find R such that

$$\partial_t R + \varepsilon^{-1} v \cdot \nabla R + \underline{G} \cdot \nabla_v R = \varepsilon^{-2} \mathcal{L} R + \varepsilon^{-1} \mathcal{L}^1 R + \mathcal{L}^2 R + D, \tag{4.15}$$

with the same initial and boundary conditions as before.

Once one obtains good estimates for the solution of this linear problem, the non linear problem is solved by simple Banach fixed point arguments, for small ε. This allows to conclude the existence of the solution f^ε and its convergence to the solution of the OBE.

Solution of Linear problem.

We consider the linear problem (4.15) with a given D satisfying $\langle v_z D \rangle = 0$. Put $R = \sqrt{\tilde{M}} \Phi$. Therefore the equation for Φ is

$$\partial_t \Phi + \varepsilon^{-1} v \cdot \nabla \Phi + \underline{G} \cdot \nabla_v \Phi = \varepsilon^{-2} \tilde{L} \Phi + \varepsilon^{-1} \tilde{L}^1 \Phi + \tilde{L}^2 \Phi + \tilde{D}, \tag{4.16}$$

where $\tilde{D} = \tilde{M}^{-1/2} D$, $\tilde{M} = M \exp[\varepsilon G(z+1)/T_-]$ and

$$\tilde{L}^j f = \tilde{M}^{-1/2} \mathcal{L}^j \tilde{M}^{1/2} f, \quad j = 1, 2$$

The boundary and initial conditions are

$$\Phi(\underline{x}, v; 0) = 0; \qquad \Phi(x, y, \pm 1, v; t) = \alpha^{\pm}_R \overline{M}_{\pm}(v) \tilde{M}^{-1/2} + \zeta^{\pm}, \qquad v_z \lessgtr 0, \ \ t > 0 \tag{4.17}$$

where

$$\zeta^{\pm} := -\tilde{M}^{-1/2} \sum_{n=2}^{7} \varepsilon^{n-3} \gamma^{\pm}_{n,\varepsilon}, \qquad \alpha^{\pm}_R = \pm \int_{v_z \gtrless 0} v_z \sqrt{\tilde{M}} \Phi(x, y, \pm 1, v; t) dv \tag{4.18}$$

Introduce the L_2 norm as

$$\| f \| = \left\{ \int_{\Omega \times \mathbb{R}^3} d\underline{x} dv |f(\underline{x}, v; t)|^2 \right\}^{1/2}$$

We now give first the L_2 bound for Φ and then we provide the L_∞ bound.

Theorem 4.2

The solution of the linear problem (4.16), (4.17) satisfy the bound

$$||\Phi|| \leq C \sup_{t\in(0,\bar{t}]} ||\tilde{D}||$$

Proof

Multiplying (4.16) by Φ and integrating on $\Omega \times \mathbb{R}^3$ we have

$$\begin{aligned}\frac{1}{2}\partial_t \parallel \Phi \parallel^2 = \varepsilon^{-2} \int_{\Omega} d\underline{x} \int_{R^3} dv \Big[\Phi\tilde{L}\Phi + \varepsilon\Phi\tilde{L}^1\Phi + \varepsilon^2\Phi\tilde{L}^2\Phi + \Phi\tilde{D}\Big] \\ + \frac{1}{2\varepsilon}\int dxdy\Big[\langle v_z\Phi^2\rangle(x,y,-1;t) - \langle v_z\Phi^2\rangle(x,y,1;t)\Big]\end{aligned} \tag{4.19}$$

To bound the boundary terms in the second line of (4.19), we proceed as follows. We consider first the more difficult term

$$\langle v_z\Phi^2\rangle(x,y,1;t) = -(\alpha_R^+)^2 \int_{v_z<0} dv|v_z|\overline{M}_+^2\tilde{M}^{-1} + \int_{v_z>0} dvv_z\Phi(x,y,1,v;t)^2. \tag{4.20}$$

By the Schwartz inequality,

$$\begin{aligned}(\alpha_R^+)^2 &= \Big(\int_{v_z>0} dv|v_z|^{\frac{1}{2}}|v_z|^{\frac{1}{2}}\tilde{M}^{\frac{1}{2}}\Phi(x,y,1,v;t)\Big)^2 \\ &\leq \int_{v_z>0} dv|v_z|\Phi^2(x,y,1,v;t)\int_{v_z>0} dv|v_z|\tilde{M}\end{aligned} \tag{4.21}$$

Using the relation between $\tilde{M}$ in $z = +1$ and $\overline{M}_+$, the normalization of $\overline{M}_+$, and the relation $T_+ = T_-(1-2\varepsilon\lambda)$, we get, after a straightforward computation,

$$\begin{aligned}&\int_{v_z>0} dv|v_z|\tilde{M}\int_{v_z>0} dvv_z\overline{M}_+^2\tilde{M}^{-1} = \Big(\frac{T_-}{2\pi}\Big)^{1/2}\int_{v_z>0} dvv_z\overline{M}_+^2M^{-1} = \\ &\Big(\frac{T_-}{2\pi}\Big)^{1/2}\frac{(2\pi T_-)^{3/2}}{(2\pi T_+^2)^2}2\pi\left(\frac{T_+T_-}{2T_- - T_+}\right)^2 = \frac{(1-2\varepsilon\lambda)^2}{(1-2\varepsilon\lambda)^2(1+2\varepsilon\lambda)^2} \\ &= (1-\varepsilon^2\lambda^2)^{-2} \leq 1 + C\varepsilon^2\end{aligned} \tag{4.22}$$

Using this bound and (4.21) in (4.20) we find

$$-\frac{1}{\varepsilon}\langle v_z\Phi^2(x,y,1,v;t)\rangle \leq C\varepsilon\int_{v_z>0} dv|v_z|\Phi^2(x,y,1,v;t) \qquad (4.23).$$

The same argument shows that $\langle v_z \Phi^2(x,y,-1,v;t)\rangle$ is non positive because

$$\int_{v_z<0} dv|v_z|\tilde{M} \int_{v_z>0} dv v_z \overline{M}_-^2 \tilde{M}^{-1} = 1.$$

In Appendix it is proved the following bound for the r.h.s. of (4.23): let t_1 be any time in $(0.\bar{t}]$. Then

$$\begin{aligned} &\varepsilon \int_0^{t_1} dt \int_{\mathbb{T}^2} dxdy \int_{v_z>0} dv|v_z|\Phi^2(x,y,1,v;t) \\ &\le \int_0^{t_1} dt \parallel \Phi(\,\cdot\,,t) \parallel^2 + \varepsilon^4 \sup_{0<t\le t_1} \parallel \tilde{D}(\,\cdot\,,t) \parallel^2 + \sup_{0<t\le t_1} |\zeta(\,\cdot\,,t)|^2 \end{aligned} \tag{4.24}$$

where $|\zeta| = |\zeta^+| + |\zeta^-|$.

We recall the following crucial properties of the linearized Boltzmann operator $\tilde{L}$ (see for example [13]):

1)

$$\tilde{L} = -\nu + K,$$

with K an integral operator and ν a smooth function. Moreover, for hard spheres, there are two constants ν_0 and ν_1 such that

$$\nu_0(1+|v|) \le \nu(x,v) \le \nu_1(1+|v|);$$

2) There is a constant $C > 0$ such that

$$\langle \Phi \tilde{L} \Phi \rangle \le -C\langle \nu \bar{\Phi}^2 \rangle. \tag{4.25}$$

with the usual decomposition $\Phi = \bar{\Phi} + \hat{\Phi}$, with $\bar{\Phi}$ and $\hat{\Phi}$ the non-hydrodynamical and the hydrodynamical part of Φ respectively.

To estimate the operators $\tilde{L}^1$ and $\tilde{L}^2$ we will use the following estimate on the collision operator Q (see for example [14]): for any Maxwellian M and for any $y \in [-1,1]$

$$\int_{\mathbb{R}^3} dv \frac{|Q(\sqrt{M}f, \sqrt{M}g)|^2}{\nu M} \le \int_{\mathbb{R}^3} dv \nu |f|^2 \int_{\mathbb{R}^3} dv \nu |g|^2 \tag{4.26}$$

This inequality and the bounds on the f_n's imply the following bounds:

$$\int dxdv\Phi\tilde{L}^1\Phi \leq C \parallel \sqrt{\nu}\bar{\Phi} \parallel \parallel \Phi \parallel\parallel M^{-1/2}f_1 \parallel, \tag{4.27}$$

$$\int dxdv\Phi\tilde{L}^2\Phi \leq C \parallel \sqrt{\nu}\bar{\Phi} \parallel \parallel \Phi \parallel\parallel M^{-1/2}\sum_{n=2}^{7} f_n \parallel . \tag{4.28}$$

Note that the presence of the product $\parallel \sqrt{\nu}\bar{\Phi} \parallel\parallel \Phi \parallel$ depends on from the fact that $\tilde{L}^1$ and $\tilde{L}^2$ are both orthogonal to the collision invariants.

We integrate (4.19) on time between 0 and t_1, and recall that $\Phi(x, v, 0) = 0$. With the notation $\Phi_t(\,\cdot\,) = \Phi(\,\cdot\,, t)$, we get

$$\begin{aligned} \frac{1}{2} \parallel \Phi_{t_1} \parallel^2 \leq C \int_0^{t_1} dt \Bigg\{ & - \varepsilon^{-2} \parallel \sqrt{\nu}\bar{\Phi}_t \parallel^2 \\ + \parallel \sqrt{\nu}\bar{\Phi}_t \parallel \parallel \Phi_t \parallel & \left[\varepsilon^{-1} \parallel M^{-1/2} f_1 \parallel + \parallel M^{-1/2} \sum_{n=2}^{7} f_n \parallel \right] + \parallel \tilde{D}_t \parallel^2 \Bigg\} \\ + C \int_0^{t_1} dt \parallel \Phi_t \parallel^2 & + \varepsilon^4 \sup_{0<t\leq t_1} \parallel \tilde{D}(\,\cdot\,, t) \parallel^2 + \sup_{0<t\leq t_1} |\zeta(\,\cdot\,, t)|^2 \end{aligned} \tag{4.29}$$

The last line in (4.29) derives from the bound (4.24). The first term in the second line is due to the bounds (4.27) and (4.28). Moreover $\parallel M^{-1/2} f_1 \parallel \leq C$ by the regularity of the solutions of the macroscopic equations for $0 < t < \bar{t}$ and $\parallel \tilde{M}^{-1/2} \sum_{n=2}^{7} f_n \parallel \leq C$ by Theorem 4.1.

The following elementary inequality

$$-\frac{1}{\varepsilon^2} x^2 + (c_1\varepsilon^{-1} + c_2)xy \leq (c_1\varepsilon + c_2)^2 y^2/4$$

is valid for any positive ε, x, y. We apply it with $x = ||\sqrt{\nu}\bar{\Phi}||$, $y = ||\Phi||$ and suitable constants c_1 and c_2. We get

$$\begin{aligned} \parallel \Phi(\,\cdot\,, t_1) \parallel^2 \leq & \int_0^{t_1} dtC \left[\parallel \Phi(\,\cdot\,, t) \parallel^2 + \parallel \tilde{D}(\,\cdot\,, t) \parallel^2 \right] \\ & + \varepsilon^4 \sup_{0<t\leq t_1} \parallel \tilde{D}(\,\cdot\,, t) \parallel^2 + \sup_{0<t\leq t_1} |\zeta(\,\cdot\,, t)|^2 \end{aligned}$$

In conclusion, by the use of the Gronwall lemma, for ε sufficiently small, we get:

$$\sup_{0\le t\le \bar t} \parallel \Phi(\,\cdot\,,t) \parallel \le C_{\bar t} \sup_{t\in(0,\bar t]} \parallel \tilde D(\,\cdot\,,t) \parallel \; + \sup_{t\in(0,\bar t]} |\zeta(\,\cdot\,,t)| \tag{4.30}$$

L_∞ *bound.*

Let us give some notation:

$$\partial\Lambda_\pm := \Big\{ \underline{x} = (x,y,z) : (x,y) \in \mathbb{T}^2, z = \pm 1 \Big\}$$

$$S := \Big\{ (t,\underline{x},v) : t \in (0,\bar t], (\underline{x},v) \in \Omega \Big\} \tag{4.31}$$

$$\partial S := \Big\{ (t,\underline{x},v) : t = 0, (\underline{x},v) \in \Omega \Big\} \cup \Big\{ (t,\underline{x},v) : t \in (0,\bar t], \underline{x} \in \partial\Lambda_\pm, v_z \lesseqgtr 0 \Big\}. \tag{4.32}$$

Λ and Ω are defined in (3.1).

We call $\varphi_t(x,v)$ the characteristics of the equation

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f + \underline{G} \cdot \nabla_v f = 0$$

given by

$$\varphi_t(x,v) = (\underline{x} + \frac{v}{\varepsilon} t + \frac{1}{2\varepsilon} \underline{G} t^2, v + \underline{G} t). \tag{4.33}$$

and define

$$t^- = \inf\{t \ge 0 : (t, \varphi_t(x,v)) \in S\}; \qquad t^+ = \sup\{t^- < t < \bar t : (s, \varphi_s(x,v)) \in S \;\forall s \le t\} \tag{4.34}$$

Given smooth functions H, $\tilde\nu$ on S and h in ∂S, consider the initial boundary value problem

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f + \underline{G} \cdot \nabla_{v_z} f - \frac{1}{\varepsilon^2} \tilde\nu f = H \tag{4.35}$$

$$f(t,\underline{x},v) = h(t,\underline{x},v), \qquad (t,\underline{x},v) \in \partial S. \tag{4.36}$$

The solution of this problem is written as

$$\begin{aligned} f(t,x,v) &= h\big(t^-,\varphi_{t^- -t}(x,v)\big)\exp\Big\{-\int_{t^-}^t ds\frac{1}{\varepsilon^2}\tilde\nu\big(\varphi_{s-t}(x,v)\big)\Big\} \\ &\quad + \int_{t^-}^t dsH\big(s,\varphi_{s-t}(x,v)\big)\exp\Big\{-\int_s^t ds'\frac{1}{\varepsilon^2}\tilde\nu\big(\varphi_{s'-t}(x,v)\big)\Big\}. \end{aligned} \tag{4.37}$$

We introduce the norm

$$||f||_{p,q} := \left[\int_{\mathbb{T}^2\times[-1,1]} d\underline{x}\Big[\int_{\mathbb{R}^3} dv|f|^q\Big]^{\frac{p}{q}}\right]^{\frac{1}{p}} \tag{4.38}$$

and denote by $L^{q,p}$ the corresponding space. We define the operator N_s^t as

$$N_s^t f = f\big(s,\varphi_{s-t}(x,v)\big)\exp\Big\{-\int_s^t ds'\frac{1}{\varepsilon^2}\tilde\nu\big(\varphi_{s-t}(x,v)\big)\Big\}. \tag{4.39}$$

We will also omit the apex t when there is no risk of confusion.

We assume that $\tilde\nu$ corresponds to the collision rate for hard spheres, i.e.

$$\nu_0(1+|v|)\le\tilde\nu\le\nu_1(1+|v|), \tag{4.40}$$

Then N_s^t satisfies the estimate

$$||N_s^t f||_{p,q}\le C\exp\{-\frac{\nu_0(t-s)}{\varepsilon^2}\}||f||_{p,q} \tag{4.41}$$

The following Lemma allows to bound the $L^{p,q}$ norm of $N_s f$ in terms of the $L^{q,p}$ norm of f. It is a generalization to the case with a constant force of the theorem of Ukai and Asano [9].

Lemma 4.3

Let $1\le q,p\le+\infty$ and $\alpha=\frac{1}{q}-\frac{1}{p}$. Then

$$||N_s^t f||_{p,q}\le\Big(\frac{\varepsilon}{t-s}\Big)^{d\alpha}\exp\{-\frac{\nu_0(t-s)}{\varepsilon^2}\}||f||_{q,p}. \tag{4.42}$$

Proof.

Consider the following change of variables $v \to \underline{y}$:

$$\underline{y} = \underline{x} + \frac{1}{\varepsilon} v\tau + \frac{1}{2\varepsilon}\underline{G}\tau^2, \qquad \text{or} \quad v + \frac{1}{2}\underline{G}\tau = \varepsilon\frac{\underline{y}-\underline{x}}{\tau} \tag{4.43}$$

where we have put $\tau = s - t$. We have under the change (4.43)

$$||N_s^t f||_{p,q} \le \left(\frac{\varepsilon}{\tau}\right)^{\frac{d}{q}} \exp\{-\frac{\nu_0\tau}{\varepsilon^2}\}\left[\int d\underline{x}\Big[\int d\underline{y}\Big|f(\tau+t,\underline{y},\varepsilon\frac{\underline{y}-\underline{x}}{\tau}+\frac{1}{2}G\tau)\Big|^q\Big]^{\frac{p}{q}}\right]^{\frac{1}{p}}$$

With one more change of variables

$$\underline{x} \to w = \varepsilon\frac{\underline{y}-\underline{x}}{\tau} + \frac{1}{2}G\tau,$$

we get

$$||N_s^t f||_{p,q} \le \left(\frac{\varepsilon}{\tau}\right)^{\frac{d}{q}}\left(\frac{\varepsilon}{\tau}\right)^{-\frac{d}{p}} \exp\{\frac{\nu_0\tau}{\varepsilon^2}\}||f||_{p,q}$$

Hence the result.

This Lemma will be used also for $p = +\infty$ and $q > d$ to control the (∞, q)-norm of $N_s^t f$ in terms of (q,∞)-norm of the solution of (4.37).

We write (4.16), (4.17) in the form (4.35), (4.36) with

$$H = \varepsilon^{-2}\tilde{K}\Phi + \varepsilon^{-1}\tilde{K}^1\Phi + \tilde{K}^2\Phi + \tilde{D} := \varepsilon^{-2}\tilde{K}\Phi + H', \tag{4.44}$$

$\tilde{L}$ is the linear Boltzmann operator such that

$$\tilde{L} := \tilde{K} - \nu,$$

$$\tilde{L}^i := \tilde{K}^i - \nu^i, \qquad i = 1, 2.$$

Here the ν^i are defined analogously to ν as

$$\nu^1(\underline{x}, v) = M^{-1/2}\int dv' d\omega (v'-v)\cdot\omega f_1\sqrt{\tilde{M}},$$

$$\nu^2(\underline{x}, v) = M^{-1/2}\int dv' d\omega (v'-v)\cdot\omega \sum_{n=2}^{7} f_n\sqrt{\tilde{M}}.$$

Finally we set $\bar{\nu} = \tilde{\nu} + \varepsilon\tilde{\nu}^1 + \varepsilon^2\tilde{\nu}^2$ and it is immediate to check that $\bar{\nu}$ satisfies the assumption (4.40).

We have the following

Theorem 4.4

Let Φ be the solution of the problem (4.35), (4.36), with $\tilde{\nu}$, H and h given as before. Then, for any $q > d$, $\frac{1}{2} - \frac{1}{d} < \frac{1}{q}$,

$$\sup_{0 \le t \le \bar{t}} \| \Phi \|_{\infty,q} \le C\varepsilon^{-d/2} \sup_{0 \le t \le \bar{t}} \| \tilde{D} \|_{2,2} + C\varepsilon^{-d/2} \sup_{0 \le t \le \bar{t}} \| \zeta \|_{2,2}$$
$$+ C\varepsilon^2 \sup_{0 \le t \le \bar{t}} \| H' \|_{\infty,q} + C\varepsilon^2 \sup_{0 \le t \le \bar{t}} \| \zeta \|_{\infty,q} . \tag{4.45}$$

Proof.

By (4.37) we get

$$\| \Phi \|_{\infty,q} \le \| N^t_{t^-} h \|_{\infty,q} + \int_{t^-}^t ds \big|\big| N^t_s H' \big|\big|_{\infty,q} + \varepsilon^{-2} \int_{t^-}^t ds \, \| N^t_s \tilde{K} \Phi \|_{\infty,q} \tag{4.46}$$

In the last term we substitute to Φ its expression (4.37) so that

$$\| N^t_s \tilde{K} \Phi \|_{\infty,q}$$
$$\le \| N^t_s \tilde{K} N^s_{s^-} h \|_{\infty,q} + \int_{s^-}^s ds' \Bigg[\| N^t_s \tilde{K} N^s_{s'} H' \|_{\infty,q} + \varepsilon^{-2} \| N^t_s \tilde{K} N^s_{s'} \tilde{K} \Phi \|_{\infty,q} \Bigg]. \tag{4.47}$$

The boundary term will be discussed separately. To estimate the terms containing $\tilde{K}$ we use the following Lemma whose proof is given in [9]

Lemma 4.5

Let L^s_γ the spaces of functions $f(v)$ such that

$$\int_{\mathbb{R}^3} dv |f(v)|^s (1 + |v|)^\gamma < \infty$$

with $\gamma \in \mathbb{R}$. Let $1 \le s \le r \le \infty$ and $\eta_0 = 1 - \frac{1}{s} + \frac{1}{r}$. Then the operator K maps L^s_γ into $L^r_{\gamma+\eta}$ if $\frac{1}{s} - \frac{1}{r} < \frac{2}{d}$ and $\eta \le \eta_0$.

We have by (4.41) and by Lemma 4.5

$$\| \Phi \|_{\infty,q} \le \sup_{0 \le t \le \bar{t}} \| N^t_{t_-} h \|_{\infty,q} + C \sup_{0 \le t \le \bar{t}} \big|\big| H' \big|\big|_{\infty,q} \int_{t_-}^t ds \exp\{-\frac{\nu_0 (t-s)}{\varepsilon^2}\}$$
$$+ \varepsilon^{-4} \int_{t^-}^t ds \int_{s^-}^s ds' \, \| N^t_s \tilde{K} N^s_{s'} \tilde{K} \Phi \|_{\infty,q} \tag{4.48}$$

Using the bound $\int_{t_-}^{t} ds \exp[-\varepsilon^{-2}\nu_0(t-s)] \leq C\varepsilon^2$, the estimate for the second term in the r.h.s. of (4.48) follows. We bound the last one by using Lemmas 4.3 and 4.5 as follows.

$$\begin{aligned}
\| N_s^t \tilde{K} N_{s'}^s \tilde{K}\Phi \|_{\infty,q} &\leq C\Big(\frac{\varepsilon}{(t-s)}\Big)^{d/q} \exp\{-\frac{\nu_0}{\varepsilon^2}(t-s)\} \| \tilde{K} N_{s'}^s \tilde{K}\Phi \|_{q,\infty} \\
&\leq C\Big(\frac{\varepsilon}{(t-s)}\Big)^{d/q} \exp\{-\frac{\nu_0}{\varepsilon^2}(t-s)\} \| N_{s'}^s \tilde{K}\Phi \|_{q,2} \\
&\leq C\Big(\frac{\varepsilon}{(t-s)}\Big)^{d/q} \Big(\frac{\varepsilon}{(s-s')}\Big)^{d\beta} \exp\{-\frac{\nu_0}{\varepsilon^2}[(t-s)+(s-s')]\} \| \tilde{K}\Phi \|_{2,q} \\
&\leq C\Big(\frac{\varepsilon}{(t-s)}\Big)^{d/q} \Big(\frac{\varepsilon}{(s-s')}\Big)^{d\beta} \exp\{-\frac{\nu_0}{\varepsilon^2}[(t-s)+(s-s')]\} \| \Phi \|_{2,2}
\end{aligned} \tag{4.49}$$

To get the first inequality we have used (4.41) and the Lemma 4.3 which hold for $q > d$. The second step is based on the Lemma 4.5 to replace the L^∞ norm on the velocity with the L^2 norm, by taking $r = +\infty$, $s = 2$, $\gamma = 0$. Then the Lemma 4.3 allows to exchange the exponents for space and velocity introducing a factor $(\varepsilon/(s-s'))^{d\beta} \exp[-\nu_0(s-s')/\varepsilon^2]$, with $\beta = \frac{1}{2} - \frac{1}{q}$. Finally the Lemma 4.5 again, with $r = q$, $s = 2$ and $\gamma = 0$, gives the bound in terms of the L^2 norm in space and velocity.

To get convergence of the s'-integral we need $0 < \beta < \frac{1}{d}$, so we have to choose $q > d$, $\frac{1}{2} - \frac{1}{d} < \frac{1}{q}$. From the time integrations we get a factor $\varepsilon^{4-2(\mu+\mu')}$ with $\mu = d/q, \mu' = d\beta$. Combined with the prefactor $\varepsilon^{(\mu+\mu')}$ it produces $\varepsilon^{4-(\mu+\mu')}$. But $\mu + \mu' = d/2$ so that the factor we gain is $\varepsilon^{4-d/2}$. In conclusion

$$\begin{aligned}
\sup_{0\leq t\leq \bar{t}} \| \Phi \|_{\infty,q} \leq C \sup_{0\leq t\leq \bar{t}} \| N_{t_-} h \|_{\infty,q} + C\varepsilon^2 \sup_{0\leq t\leq \bar{t}} \| H' \|_{\infty,q} \\
+ C\varepsilon^{-d/2} \sup_{0\leq t\leq \bar{t}} \| \Phi \|_{2,2}
\end{aligned} \tag{4.50}$$

Now we bound the boundary term. We recall that its meaning is as follows: define

$$\tau_- = \inf\{\tau > 0 : \varphi_{-\tau}(\underline{x}, v; t) \in \partial S\}$$

If $\tau_- = t$ then $h = \Phi(\underline{x}, v; 0) = 0$; if $\tau_- < t$ then $h = \Phi(x, y, \pm 1, v; t)$, in correspondence of $v_z \lessgtr 0$ and $\big(\varphi_{-\tau}(\underline{x}, v; t)\big)_z = \pm 1$. The boundary conditions on Φ are given by (4.17) with

$$\alpha_R^{\pm} = \pm \int_{v_z \gtrless 0} v_z \sqrt{\tilde{M}}\Phi(x, y, \pm 1, v; t) dv$$

Equation (4.37) allows to express α_R^+ in terms of $\Phi(x,y,-1,v,t), v_z < 0$ and H. In fact by (4.37)

$$\begin{aligned} \Phi(x,y,1,v;t) &= h\big(t^-, \varphi_{t^- - t}(x,y,1,v)\big) \exp\Big\{ - \int_{t^-}^t ds \frac{1}{\varepsilon^2} \nu\big(\varphi_{s-t}(x,y,1,v)\big)\Big\} \\ &+ \int_{t^-}^t ds H\big(s, \varphi_{s-t}(x,y,1,v)\big) \exp\Big\{ - \int_s^t ds' \frac{1}{\varepsilon^2} \nu\big(\varphi_{s'-t}(x,y,1,v)\big)\Big\}. \end{aligned} \tag{4.51}$$

We remark that the characteristic φ which is on the boundary $z = 1$ with velocity v such that $v_z > 0$ at time $t^- - t$ had to start either from some point in the bulk or from the boundary $z = -1$. In the first case we have $h = 0$ in (4.51) otherwise we get $h = \Phi(x,y,-1,v;t)$ for $v_z > 0$. In the latter case it is shown in the Appendix that the integral $\int_{t^-}^t ds \nu\big(\varphi_{s-t}(x,y,-1,v)\big)$ is bounded from below by $C\varepsilon$. As a consequence, we can estimate the exponential in the first row of (4.51) as $e^{-C/\varepsilon}$.

We exploit the same argument to deal with α_R^- and use (4.37) to represent Φ on the boundary $z = -1$. There is a difference with respect to the previous case: due to the presence of the force the characteristic φ which is on the boundary $z = -1$ with velocity $v, v_z < 0$ at time $t^- - t$ can start from the bulk, from the boundary $z = 1$ with negative v_z or from the boundary $z = -1$ with positive v_z. In the Appendix it is shown that in the latter case the exponential factor in the first row of (4.51) allows to gain a factor $C\varepsilon^2$.

We discuss explicitely the bound for α_R^-. The case of α_R^+ is dealt with in the same way. By using the representation (4.37) we get

$$\begin{aligned} &\sup_{(x,y)\in\mathbb{T}^2} |\alpha_R^-(x,y,-1;t)| = \sup_{(x,y)\in\mathbb{T}^2} \Big| \int_{v_z>0} v_z \sqrt{\widetilde{M}} \Phi(x,y,-1,v;t) dv \Big| \\ &\le C \sup_{(x,y)\in\mathbb{T}^2} \Big[\int_{v_z>0} dv |\Phi|^q(x,y,-1,v;t) \Big]^{1/q} \le \\ &C \sup_{(x,y)\in\mathbb{T}^2} [e^{-\frac{C}{\varepsilon}} |\alpha_R^+| + \varepsilon^2 |\alpha_R^-|] + \varepsilon^2 \parallel \zeta \parallel_q + \int_{t^-}^t ds \big|\big| N_s^t H \big|\big|_{\infty,q} \le \\ &C\varepsilon^2 \parallel \Phi \big|\big|_{\infty,q} + \int_{t^-}^t ds \big|\big| N_s^t H' \big|\big|_{\infty,q} + \varepsilon^{-2} \int_{t^-}^t ds \parallel N_s^t \tilde{K} \Phi \parallel_{\infty,q} \end{aligned} \tag{4.52}$$

The first bound is obtained from the inequality

$$\sup_{(x,y)\in\mathbb{T}^2}|\alpha_R^{\pm}| \le \sup_{\underline{x}\in\mathbb{T}^2\times[-1,1]}\int_{v_z\gtrless 0} v_z\sqrt{\tilde{M}}|\Phi(\underline{x},v;t)|dv$$
$$\le C\sup_{\underline{x}\in\mathbb{T}^2\times[-1,1]}\Big[\int_{v_z>0}dv|\Phi|^q(\underline{x},v;t)\Big]^{1/q} = \parallel \Phi \parallel_{\infty,q}$$

As explained before, to get a bound of the last term in (4.52) in terms of the L_2 norm we need to iterate the procedure and use again the representation formula (4.37) in the last term

$$\varepsilon^{-2}\int_{t^-}^{t} ds \parallel N_s^t\tilde{K}\Phi \parallel_{\infty,q} \le \varepsilon^{-2}\int_{t^-}^{t} ds \parallel N_s^t\tilde{K}N_{s^-}^s h \parallel_{\infty,q} +$$
$$\varepsilon^{-2}\int_{t^-}^{t} ds\int_{s^-}^{s} ds'\Big[\parallel N_s^t\tilde{K}N_{s'}^s H' \parallel_{\infty,q} + \varepsilon^{-2}\parallel N_s^t\tilde{K}N_{s'}^s\tilde{K}\Phi \parallel_{\infty,q}\Big] \tag{4.53}$$

In this way all the terms we got in (4.53) are analogous to terms already discussed in the first part of the proof but the term containing h. The problem with this term is that it can be estimated in terms of the L_∞ norm of Φ but we need to gain a small factor and $N_{s^-}^s$ cannot provide it. Hence we have to repeat the previous argument and use (4.37) to represent Φ on the boundary in terms of the function evaluated on the point on the boundary reached after a finite amount of time.

We get

$$\varepsilon^{-2}\int_{t^-}^{t} ds \parallel N_s^t\tilde{K}N_{s^-}^s h \parallel_{\infty,q} \le \varepsilon^2 \parallel \zeta \parallel_q +$$
$$\varepsilon^{-2}\int_{t^-}^{t} ds \parallel N_s^t\tilde{K}N_{s^-}^s\,[\tilde{M}^{-1/2}[\overline{M}^+\alpha_R^+ + \alpha_R^-\overline{M}^-]] \parallel_{\infty,q} + \tag{4.54}$$
$$\varepsilon^{-2}\int_{t^-}^{t} ds\int_{s^-}^{s} d\tau\Big[\parallel N_s^t\tilde{K}N_{s^-}^s N_{s\_}^\tau H' \parallel_{\infty,q} + \varepsilon^{-2}\parallel N_s^t\tilde{K}N_{s^-}^s N_{s\_}^\tau\tilde{K}\Phi \parallel_{\infty,q}\Big]$$

By using the properties of N_s and $\tilde{K}$ and the bounds on the integral in the exponential as discussed before the second term in (4.54) is bounded by $\varepsilon^2 \parallel \Phi||_{\infty,q}$.

Finally we get

$$\sup_{(x,y)\in\mathbb{T}^2}|\alpha_R^{\pm}| \le C\varepsilon^2 \parallel \Phi\big|\big|_{\infty,q} + \varepsilon^2 \parallel \zeta \parallel_q + \int_{t^-}^t ds\big|\big|N_s H'\big|\big|_{\infty,q} + \\ \varepsilon^{-2}\int_{t^-}^t ds \int_{s^-}^s ds' \Bigg[\parallel N_s^t \tilde{K} N_{s'}^s H' \parallel_{\infty,q} + \varepsilon^{-2} \parallel N_s^t \tilde{K} N_{s'}^s \tilde{K}\Phi \parallel_{\infty,q} \Bigg] \tag{4.55}$$

Now we can apply all the previous argument to get the following estimate for the boundary term in (4.50)

$$\big|\big|N_{t^-}^t h \parallel_{\infty,q} \le \sup_{(x,y)\in\mathbb{T}^2}[|\alpha_R^+| + |\alpha_R^-|] \le \\ C\varepsilon^2 \parallel \zeta \parallel_q + C\varepsilon^2 \sup_t \parallel H' \parallel_{\infty,q} \ + C\varepsilon^{-d/2} \sup_t \parallel \Phi \parallel_{2,2} \tag{4.56}$$

Finally by (4.30) we have

$$\sup_{0\le t\le \bar{t}} \parallel \Phi \parallel_{\infty,q} \le C\varepsilon^2 \sup_{0\le t\le \bar{t}} \parallel \zeta \parallel_q + C\varepsilon^2 \sup_{0\le t\le \bar{t}} \parallel H' \parallel_{\infty,q} \\ + C\varepsilon^{-d/2} \sup_{0\le t\le \bar{t}} \parallel \tilde{D} \parallel_{2,2} + \varepsilon^{-d/2} \sup_{0\le t\le \bar{t}} \parallel \zeta \parallel_2 \tag{4.57}$$

This concludes the proof of Theorem 4.4.

To get the L^∞ bound for Φ we need to estimate the $\parallel \cdot \parallel_{\infty,\infty}$ norm in terms of the $\parallel \cdot \parallel_{2,2}$ of Φ. The use Lemma 4.5 allows to prove the following

Theorem 4.6

Define

$$|f|_r = \sup_{x\in\Lambda}\sup_{v\in\mathbb{R}^3} |(1+|v|)^r f|.$$

Let Φ be the solution of (4.16), (4.17). Then if $r > 3$

$$\sup_{0\le t\le \bar{t}} |\Phi|_r \le C\varepsilon^{-d/2}\left(\sup_{0\le t\le \bar{t}} |\tilde{D}|_r + \sup_{0\le t\le \bar{t}} |\zeta|_r \right)$$

Proof

By (4.37) we have

$$|\Phi|_0 \le |N_{t^-} h|_0 + \int_{t^-}^{t} ds |N_s H'|_0 + \varepsilon^{-2} \int_{t^-}^{t} ds |N_s \tilde{K} \Phi|_0$$

By (4.41) and Lemma 4.5 (used with $r = +\infty$ and s replaced by q), for $q > \frac{d}{2}$ we have

$$|\Phi|_0 \le C\varepsilon^2 \sup_{0\le t\le \bar{t}} |h|_0 + C \sup_{0\le t\le \bar{t}} |H'|_0 + C \sup_{0\le t\le \bar{t}} |\Phi|_{\infty,q}$$

By (4.57) we get

$$|\Phi|_0 \le C \sup_{0\le t\le \bar{t}} |h|_0 + C\varepsilon^2 \left(\sup_{0\le t\le \bar{t}} |H'|_0 + \sup_{0\le t\le \bar{t}} \| H' \|_{\infty,q} + C\varepsilon^2 \sup_{0\le t\le \bar{t}} \| \zeta \|_{\infty,q} \right)$$
$$+ C\varepsilon^{-d/2} \left(\sup_{0\le t\le \bar{t}} \| \tilde{D} \|_{2,2} + \sup_{0\le t\le \bar{t}} \| \zeta \|_2 \right)$$

Because of the factor $(1+|v|)^r$ in the definition of $|\cdot|_r$ and of the boundedness of the space domain, if $r > 3$ we have

$$\| f \|_{\infty,q} \le C|f|_r, \qquad \| f \|_{2,2} \le C|f|_r.$$

Hence

$$\sup_{0\le t\le \bar{t}} |\Phi|_0 \le C \sup_{0\le t\le \bar{t}} |h|_0 + C\varepsilon^{-d/2} \left(\sup_{0\le t\le \bar{t}} |\tilde{D}|_r + \sup_{0\le t\le \bar{t}} |\zeta|_r \right) + C\varepsilon^2 \sup_{0\le t\le \bar{t}} |H'|_r. \qquad (4.58)$$

The boundary term $|h|_0 \le C|\alpha_R^+|_\infty + |\alpha_R^-|_\infty + |\zeta|_\infty$ has been estimated before by using (4.56). The term containing H' will be bounded using the Grad estimate

$$|M^{-1/2} Q^+(f,g)|_r \le C|M^{-1/2} f|_r |M^{-1/2} g|_r, \qquad (4.59)$$

where Q^+ is the gain term of the collision operator.

By using (4.59) and the definitions of $\tilde{K}^i, i = 1,2$ we get

$$\| \tilde{K}^1 \Phi \|_r \le C \| M^{-1/2} f_1 \|_r \| \Phi \|_r$$

$$\| \tilde{K}^2 \Phi \|_r \leq C \| M^{-1/2} \sum_{n=2}^{7} f_n \|_r \| \Phi \|_r \ .$$

Finally the regularity property of the hydrodynamic solution and Theorem 4.1 give

$$\| \tilde{K}^1 \Phi \|_r \leq C \| \Phi \|_r, \qquad \| \tilde{K}^2 \Phi \|_r \leq C \| \Phi \|_r \tag{4.60}$$

Hence, for ε small

$$\sup_{0 \leq t \leq \bar{t}} |\Phi|_0 \leq C \varepsilon^{-d/2} \left(\sup_{0 \leq t \leq \bar{t}} |\tilde{D}|_r + \sup_{0 \leq t \leq \bar{t}} |\zeta|_r \right). \tag{4.61}$$

Finally we improve the $| \cdot |_0$ norm to $| \cdot |_r$ norm by means of the Grad estimate and the representation (4.37) of Φ

$$|\Phi|_r \leq C \left(\varepsilon^2 \sup_{0 \leq t \leq \bar{t}} |H'|_r + \sup_{0 \leq t \leq \bar{t}} |\zeta|_r + \sup_{0 \leq t \leq \bar{t}} |\Phi|_{r-1} \right)$$

By (4.59), iterating the previous inequality we get, for ε small,

$$\sup_t |\Phi|_r \leq C \varepsilon^2 \sup_{0 \leq t \leq \bar{t}} |\tilde{D}|_r + C \sup_{0 \leq t \leq \bar{t}} |\Phi|_0 + C \sup_{0 \leq t \leq \bar{t}} |\zeta|_r. \tag{4.62}$$

By putting together (4.61) and (4.62) we eventually get

$$\sup_{0 \leq t \leq \bar{t}} |\Phi|_r \leq C \varepsilon^{-d/2} \left(\sup_{0 \leq t \leq \bar{t}} |\tilde{D}|_r + C \sup_{0 \leq t \leq \bar{t}} |\zeta|_r \right).$$

so proving the Theorem.

Non linear case

We conclude our discussion by proving the following

Theorem 4.7

There is ε_0 such that, if $\varepsilon < \varepsilon_0$, for $0 < t \leq \bar{t}$ there is a unique solution to the initial boundary value problem (3.15), (3.20), (3.21) verifying the following: for any positive integer ℓ there is a constant $c > 0$ such that

$$|R|_{\ell,h} \leq C \varepsilon \tag{4.63}$$

for any $h \leq 1/(4T_-)$.

Proof

The equation we have to solve is

$$\partial_t \Phi + \varepsilon^{-1} v \cdot \nabla \Phi + \underline{G} \cdot \nabla_v \Phi = \varepsilon^{-2} \tilde{L} \Phi + \varepsilon^{-1} \tilde{L}^1 \Phi + \tilde{L}^2 \Phi + \tilde{D}$$

with

$$\tilde{D} = \varepsilon^2 M^{-1/2} Q(\sqrt{M}\Phi, \sqrt{M}\Phi) + \varepsilon^2 M^{-1/2} A$$

and A given by (3.17).

By Theorem 4.6, for $d = 3$ we have that

$$\sup_{0 \leq t \leq \bar{t}} |\Phi|_r \leq \varepsilon^{1/2} C \left(\sup_{0 \leq t \leq \bar{t}} |\tilde{M}^{-1/2} Q(\sqrt{\tilde{M}}\Phi, \sqrt{\tilde{M}}\Phi)|_r + \sup_{0 \leq t \leq \bar{t}} |\tilde{M}^{-1/2} A|_r \right)$$
$$+ \varepsilon^{-3/2} C \sup_{0 \leq t \leq \bar{t}} |\zeta|_r$$

By (4.12)

$$\sup_{0 \leq t \leq \bar{t}} |\Phi|_r \leq C\varepsilon^{1/2} \left((\sup_{0 \leq t \leq \bar{t}} |\Phi|_r)^2 + \varepsilon \sup_{1/2} |\tilde{M}^{-1/2} A|_r \right) + C\varepsilon^{-3/2} \sup_{1/2} |\zeta|_r \tag{4.64}$$

By the arguments in [7] we then get

$$\sup_t |\Phi|_r \leq \varepsilon^{1/2} C \sup_{0 \leq t \leq \bar{t}} |\tilde{M}^{-1/2} A|_r + C\varepsilon^{-3/2} \sup_{0 \leq t \leq \bar{t}} |\zeta|_r$$

The term ζ decays exponentially fast in ε. Hence, by Theorem 4.1 the estimate (4.63) follows.

5. The stationary case

The main difference between the results for the time dependent case of previous Section and those for the stationary case we are going to present is in the restriction to small values of the Rayleigh number, we need to deal with the stationary problem. Therefore, at the

hydrodynamical level we are confined to the purely conductive solution. We hope to be able to extend our method to the convective solutions which appear for larger values of the Rayleigh number. The proof follows by argument quite similar to those presented in [7], [8] to which we refer the reader for more details. In this section we only give a sketch of the proof.

We start by recalling the stationary setup. We look for one-dimensional solutions, namely for solutions not depending on x and y so that the equation we consider is

$$v_z \partial_z f^\varepsilon + \varepsilon \underline{G} \cdot \nabla_v f^\varepsilon = \varepsilon^{-1} Q(f^\varepsilon, f^\varepsilon) \tag{5.1}$$

The boundary conditions are:

$$f^\varepsilon(-1, v) = \alpha_- \overline{M}_-(v), \quad v_z > 0, \tag{5.2}$$

$$f^e(1, v) = \alpha_+ \overline{M}_+(v), \quad v_z < 0, \tag{5.3}$$

with $\overline{M}_\pm(v)$ given by (3.5) and $\alpha_\pm$ now independent on x, y and t, given by (3.7), so that f^ε satisfies (3.6).

We construct the solution in the form (3.8) with f_n to be determined according to a bulk and boundary layer expansion. These terms are computed as in the time-dependent case and a theorem similar to Theorem 4.1 can be stated also in this case. We only discuss the remainder equation because its solution requires a different technique. This equation has the form

$$v_z \frac{\partial}{\partial z} R - \varepsilon G \frac{\partial}{\partial v_z} R = \frac{1}{\varepsilon} \mathcal{L} R + \mathcal{L}^{(1)} R + \varepsilon \mathcal{L}^{(2)} R + \varepsilon^3 Q(R, R) + \varepsilon^3 A \tag{5.4}$$

with $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)} R$ defined in (3.16). Moreover, A is given by

$$\begin{aligned} A = -v_z \frac{\partial}{\partial z} B_7 + G \frac{\partial}{\partial v_z}(B_6 + \varepsilon B_7) + (G^0 + G^-) \frac{\partial}{\partial v_z}[(b_6^+ + \varepsilon b_7^+)] + \\ (G^0 + G^+) \frac{\partial}{\partial v_z}[(b_6^- + \varepsilon b_7^-)] + \sum_{\substack{k,m \geq 1 \\ k+m \geq 8}} \varepsilon^{k+m-8} Q(f_k, f_m). \end{aligned} \tag{5.5}$$

The boundary conditions on R are given by (3.20), (3.21). R satisfies the normalization condition (3.22) and the vanishing flux condition

$$\int dv v_z R(z, v) = 0 \qquad \text{for } z \in [-1, 1]. \tag{5.6}$$

The theorem below summarizes the results about the existence of stationary solutions.

Theorem 5.1

Let M be the Maxwellian with parameters $\bar{\rho}$, T_- and vanishing mean velocity. Put

$$f_1 = M\Big(\frac{\hat{r}}{\bar{\rho}} + \frac{v^2 - 3T_-}{2T_-^2}\hat{\theta}\Big) \tag{5.7}$$

with $\hat{r}$ and $\hat{\theta}$ the thermal conduction solution of the OBE corresponding to the temperatures T_- and $T_+ = T_-(1 - 2\varepsilon\lambda)$, namely

$$\hat{\theta} = T_-\lambda(1+z), \qquad \hat{r} = -\bar{\rho}[\frac{G}{T_-} - \lambda]z.$$

Then there are $\lambda_0 > 0$ and $\varepsilon_0 > 0$ such that, if $\lambda < \lambda_0$ and $\varepsilon < \varepsilon_0$, there is a stationary solution to the boundary value problem above such that

$$|f^\varepsilon - (M + \varepsilon f_1)|_r \leq C\varepsilon^2 \tag{5.8}$$

Sketch of the proof.

We follow the strategy of the previous section: first we get an L_2 bound and then the L_∞ bound. In the present case we cannot use the initial condition to satisfy the normalization condition. Therefore, we satisfy the conditions (3.22) and (5.6) by choosing the constants $\alpha_R^\pm$ along the lines of [8].

Observe that (5.6) is satisfied for any $z \in [-1, 1]$, once it is satisfied at one point, because $\int dvv_z R(z, v)$ does not depend on z in consequence of (5.4).

We write R as

$$R = I(\Phi)\tilde{M} + \sqrt{\tilde{M}}\Phi \tag{5.9}$$

with

$$I(\Phi) = -\int_{-1}^{1} dz \int_{\mathbb{R}^3} dvR(z, v), \tag{5.10}$$

so that (3.22) is satisfied. Therefore we have $\alpha_R^- = (T_-/2\pi)^{-1/2}\bar{\rho}^{-1}I(\Phi)$. It is easy to

check (see [8]) that the function Φ has to solve the following boundary value problem:

$$\begin{aligned}
&v_z \frac{\partial \Phi}{\partial z} - \varepsilon G \frac{\partial \Phi}{\partial v_z} = \frac{1}{\varepsilon}\tilde{L}\Phi + \tilde{L}^1\Phi + \mathcal{N}\Phi + \varepsilon^2 \tilde{Q}(\Phi,\Phi) + \varepsilon^2 A,\\
&\Phi(-1,v) = \tilde{M}^{-1/2}\zeta^- \quad v_z > 0,\\
&\Phi(1,v) = \beta_R \overline{M}_+(v)\tilde{M}^{-1/2} + \zeta^+\tilde{M}^{-1/2} \quad v_z < 0\\
&\int dv v_z \Phi \tilde{M}^{1/2} = 0
\end{aligned} \tag{5.11}$$

where $\zeta^\pm = -\sum_{n=1}^6 \varepsilon^{n-3}\gamma^\pm_{n,\varepsilon}$ and $\beta_R = \alpha^+_R - \alpha^-_R$. The linear operator $\mathcal{N}\Phi$ is defined by

$$\mathcal{N}\Phi = \varepsilon L^2\Phi + I(\sqrt{\tilde{M}}\Phi)\Big[\sum_{n=2}^6 \varepsilon^n \tilde{L} f_n\Big]. \tag{5.12}$$

The non linear term is given by

$$\tilde{Q}(\Phi,\Phi) = \frac{1}{\sqrt{M}} Q(\sqrt{M}\Phi, \sqrt{M}\Phi) + 2I(\sqrt{\tilde{M}}\Phi)\tilde{L}\Phi. \tag{5.13}$$

In this way there is no normalization condition on the function Φ. The quantity α^-_R represents both the outgoing flux of f_R at $z=-1$ and the integral of R over z and v. The constant β_R is determined so that R satisfies condition (5.6) at the point $z=1$, i.e.

$$\beta_R = \int_{v_z>0} dv v_z R(1,v) + \int_{v_z<0} dv v_z \zeta^+. \tag{5.14}$$

To construct the solution of (5.11), we first consider the following linear boundary value problem: given D on $[-1,1]\times\mathbb{R}^3$ and $\zeta^\pm$ on $\{v \in \mathbb{R}^3 s.t.\, v_z \lesseqgtr 0\}$, find R satisfying

$$v_z \frac{\partial \Phi}{\partial z} - \varepsilon G \frac{\partial \Phi}{\partial v_z} = \frac{1}{\varepsilon}\tilde{L}\Phi + \tilde{L}^1\Phi + \mathcal{N}\Phi + \tilde{D}, \tag{5.15}$$

and the last three conditions (5.11).

As usual we introduce $\hat{\Phi}$ and $\bar{\Phi}$ the hydrodynamic and the non-hydrodynamic part of Φ respectively. Multiplying (5.15) by Φ and integrating on velocities, we have

$$\frac{1}{2}\langle v_z \Phi^2\rangle = \frac{1}{\varepsilon}\langle \bar{\Phi}\tilde{L}\bar{\Phi}\rangle + \langle \bar{\Phi}\tilde{L}^1\Phi\rangle + \langle \bar{\Phi}\mathcal{N}\Phi\rangle + \langle \tilde{D}\Phi\rangle$$

By integrating over z, using (4.25), (4.27) and (4.28) we get

$$\frac{1}{2}\langle v_z\Phi^2\rangle(1) - \frac{1}{2}\langle v_z\Phi^2\rangle(-1) \le -C\frac{1}{\varepsilon} \parallel \bar{\Phi} \parallel^2 + C[\lambda+\varepsilon] \parallel \bar{\Phi} \parallel \ [\parallel \hat{\Phi} \parallel + \parallel \bar{\Phi} \parallel] + \parallel \tilde{D} \parallel \parallel \Phi \parallel .$$

Here $\| \Phi \|^2 = \int_{-1}^{1} dz \int dv \phi^2 \nu$. One can check (see [8]) that the l.h.s. of this inequality is positive. Therefore we get

$$\| \bar{\Phi} \|^2 \leq C\varepsilon\lambda \| \bar{\Phi} \| \| \hat{\Phi} \| + \varepsilon \| \tilde{D} \| \| \Phi \|$$

Using the inequality $xy \leq kx^2 + y^2/4k$, with $x = \| \bar{\Phi} \|$, $y = \varepsilon \| \hat{\Phi} \|$ and a suitably small k, we find

$$\| \bar{\Phi} \|^2 \leq C\varepsilon^2\lambda \| \hat{\Phi} \|^2 + \varepsilon \| \tilde{D} \| \| \hat{\Phi} \| + C\varepsilon^2 \| \tilde{D} \|^2 \tag{5.16}$$

To bound the hydrodynamical part, we multiply (5.15) by $v_z \Psi_i$, $\Psi_i = \sqrt{\tilde{M}} \chi_i$, $i \neq 2$ and integrate over $[-1, z] \times \mathbb{R}^3$.

Denoting $p_i(z) = < v_z \Psi_i \Phi >$, we get

$$p_i(z) = p_i(-1) + \int_{-1}^{z} dz' \int dv v_z \Psi_i \Big[\frac{1}{\varepsilon} \tilde{L} \bar{\Phi} + \tilde{L}^1 \Phi + \varepsilon \mathcal{N} \Phi + \varepsilon G \frac{\partial}{\partial v_z} \Phi \Big]$$

Following [7], pag.68–69, we can prove that p_i have the following estimate:

$$|p_i(-1)| \leq \big[C\lambda \| \bar{\Phi} \| \| \hat{\Phi} \| + \frac{1}{\varepsilon} \| \tilde{D} \| \| \Phi \| \big]^{\frac{1}{2}}.$$

Using (5.16) this inequality becomes

$$p_i(z) = \big[C\lambda \| \bar{\Phi} \| \| \hat{\Phi} \| + \frac{1}{\varepsilon} \| \tilde{D} \| \| \Phi \| \big]^{\frac{1}{2}} + C \Big[\frac{1}{\varepsilon} \| \bar{\Phi} \| + (\lambda + \varepsilon) \| \hat{\Phi} \| \Big] \tag{5.17}$$

Since $\langle v_z \Phi \sqrt{\tilde{M}} \rangle = 0$, we can decompose $\hat{\Phi}$ as

$$\Phi = \sum_{i \neq 2} h_i \Psi_i.$$

Therefore

$$p_i(z) = \sum_{i \neq 2} h_j B_{ij} + < v_z^2 \Psi_i \bar{\Phi} >$$

with B a non-singular matrix. This allows to estimate h_i and hence $\hat{\Phi}$ as

$$\| \hat{\Phi} \| \leq C \Big[\frac{1}{\varepsilon} \| \bar{\Phi} \| + \frac{1}{\varepsilon} \| D \| + \lambda \| \hat{\Phi} \| \Big] \tag{5.18}$$

so that for λ small enough we get

$$\| \hat{\Phi} \| \leq C\Big[\frac{1}{\varepsilon} \| \bar{\Phi} \| + \frac{1}{\varepsilon} \| D \| \Big] \tag{5.19}$$

Combining (5.16) and (5.19) we get

$$\| \bar{\Phi} \| \leq C \| D \|, \qquad \| \hat{\Phi} \| \leq C\frac{1}{\varepsilon} \| D \| \tag{5.20}$$

Note that the only point where we need λ small is to pass from (5.18) to (5.19).

L^∞ bounds.

To write the integral form of the linear equation (5.15) we follow the approach in [11]. The notation is as follows:
The "total energy" is $E(z,v) = v_z^2/2 + V(z)$ with $V(z) = \varepsilon G(z+1)$. The lines with fixed E are the characteristic curves of the equation

$$v_z \frac{\partial f}{\partial z} - \varepsilon G \frac{\partial f}{\partial v_z} = 0$$

For $E(z,v) > V(z')$ we define

$$v_3(v,z,z') = \sqrt{2E(z,v) - 2V(z')}$$

$$v(z,z') = (v_x, v_y, v_3(v,z,z'))$$

$$\mathcal{R}v = (v_x, v_y, -v_z); \quad \mathcal{R}v(z,z') = (v_x, v_y, -v_3(v,x,x'))$$

Moreover call $z^+(z,v)$ the function implicitly defined by the equation

$$E(z,v) = V(z^+)$$

Finally put

$$\Omega_{z,z'}(v) = \int_{z'}^{z} d\eta \frac{\nu(\eta, v(z,\eta))}{v_3(v,z,\eta)}; \quad \mathcal{R}\Omega_{z,z'}(v) = \int_{z'}^{z} d\eta \frac{\nu(\eta, Rv(z,\eta))}{v_3(v,z,\eta)}$$

Consider the equation

$$v_z \frac{\partial \Phi}{\partial z} - \varepsilon G \frac{\partial \Phi}{\partial v_z} + \frac{1}{\varepsilon}\tilde{\nu}\Phi = \frac{1}{\varepsilon} H \tag{5.21}$$

with boundary conditions

$$f(\pm 1, v) = h(v)^{\pm}, \quad v_z \gtrless 0$$

The solution of (5.21) can be written in an integral form as:

for $v_z > 0$:

$$\begin{aligned} f(z,v) &= h^-(v(z,-1)) \exp -\frac{1}{\varepsilon}\Omega_{z,-1}(v) \\ &\quad + \int_{-1}^{z} dz' \frac{1}{\varepsilon} \frac{1}{v_3(v,z,z')} H(z', v(z,z')) \exp -\frac{1}{\varepsilon}\Omega_{z,z'}(v), \end{aligned} \tag{5.22}$$

for $v_z < 0$ and $E < V(1)$:

$$\begin{aligned} f(z,v) = h^-(v(z,-1)) &\exp -[\frac{1}{\varepsilon}\Omega_{z^+,-1} + \frac{1}{\varepsilon}\mathcal{R}\Omega_{z^+,z}] \\ + \Big[\int_0^{z^+} & dz' \frac{1}{v_3(v,z,z')} H(z', (v(z,z'))) \exp -[\frac{1}{\varepsilon}\Omega_{z^+,z'} + \frac{1}{\varepsilon}\mathcal{R}\Omega_{z^+,z}] \\ + \int_z^{z^+} & dz' \frac{1}{\varepsilon} \frac{1}{v_3(v,z,z')} H(z', Rv(z,z')) \exp -\frac{1}{\varepsilon}\mathcal{R}\Omega_{z,z'}\Big], \end{aligned} \tag{5.23}$$

for $v_z < 0$ and $E > V(1)$:

$$\begin{aligned} f(z,v) &= \exp -\frac{1}{\varepsilon}\Omega_{1,z}(v) h^+(v(z,1)) \\ &\quad + \int_z^1 dz' \frac{1}{\varepsilon} \frac{1}{v_3(v,z,z')} H(z', v(z,z')) \exp -\frac{1}{\varepsilon}\Omega_{z',z}. \end{aligned} \tag{5.24}$$

We can write the previous formulas in a compact form as

$$f = Vh + TH \tag{5.25}$$

where

$$\begin{aligned} V_1 f(z,v) &= \chi(v_z > 0) h^-(v(z,-1)) \exp -\frac{1}{\varepsilon}\Omega_{z,-1}(v), \\ V_2 f(z,v) &= \chi(v_z < 0)\chi(E < V(1)) h^-(v(z,-1)) \exp -\frac{1}{\varepsilon}[\Omega_{z^+,-1} + \mathcal{R}\Omega_{z^+,z}], \\ V_3 f(z,v) &= \chi(v_z < 0)\chi(E > V(1)) h^+(v(z,1)) \exp -\frac{1}{\varepsilon}\Omega_{1,z} \\ Vf &= \sum_{i=1}^{3} V_i f. \end{aligned}$$

The definition of T is given in a similar way.

By slightly modifying the proof in [11] to take into account the factor ε it is possible to prove the following Lemmas (see also [7]).

Lemma 5.2

For any integer $r \geq 0$ there is a constant c such that the integral operator T satisfies the following inequality,

$$|T\ H|_r \leq c\left|\frac{H}{\nu}\right|_r. \tag{5.26}$$

Here

$$|f|_r = \sup_{z,v}(1+|v|)^r|f(z,v)|.$$

Lemma 5.3

For any $\delta > 0$ and for any $r \geq 2$ there is a constant C_δ such that

$$N(T\ H) \leq \frac{1}{\sqrt{\varepsilon}}C_\delta \parallel \nu^{-1/2}H \parallel +\delta|H|_r. \tag{5.27}$$

Here

$$N(f) = \sup_{z\in[-1,1]}\left(\int_{\mathbb{R}^3} dv|f(z,v)|^2\right)^{1/2}.$$

We write (5.14) in the form (5.21) with

$$H = \tilde{K}\Phi + \varepsilon\tilde{L}\Phi + \varepsilon\tilde{L}^1\Phi + \varepsilon\mathcal{N}\Phi + \varepsilon D \tag{5.28}$$

and $h^{\pm} = \Phi(\pm 1, v)$ given by (5.11). Combining these Lemmas and using the properties of the operator L one gets

Theorem 5.4

There exists a constant C such that for any $r \geq 3$ the solution of (5.15) verifies

$$|\Phi|_r \leq C\frac{1}{\sqrt{\varepsilon}} \parallel \nu^{-1/2}\Phi \parallel_2) + \frac{1}{\sqrt{\varepsilon}} \parallel \varepsilon\nu^{-1/2}\tilde{D} \parallel +\varepsilon|\tilde{D}|_r + |Vh|_r \tag{5.29}$$

Noting that $\| \nu^{-1/2} D \| \leq |\nu^{-1}\tilde{D}|_3$ and using (5.19) we get

$$|\Phi|_r \leq C \frac{1}{\sqrt{\varepsilon}} |D|_{r-1} + |Ah|_r \tag{5.30}$$

Finally (5.30) implies

$$|R\tilde{M}^{-1/2}|_r \leq C\sqrt{\varepsilon}|\tilde{M}^{-1/2}D|_{r-1} + |Vh|_r \tag{5.31}$$

The term containing h involves β_R which still depends on Φ. To estimate it, we follow the method in [8].

Equation (5.25) allows to express β_R in terms of TH and the restriction of $\Phi(-1, v)$ to $v_z > 0$. We have the estimate

$$\begin{aligned} |\int_{v_z>0} v_z M_*^{1/2} h(1,v)| = |\int_{v_z>0} v_z M_*^{1/2} \{ & h^-(v(z,-1)) \exp -\frac{1}{\varepsilon}\Omega_{z,-1}(v) \\ & + \int_{-1}^{z} dz' \frac{1}{\varepsilon} \frac{1}{v_3(v,z,z')} H(z', v(z,z')) \exp -\frac{1}{\varepsilon}\Omega_{z,z'}(v), \}| \\ & \leq |s| + |\int_{v_z>0} v_z M^{1/2} TH|, \end{aligned} \tag{5.32}$$

with $s_\pm = \zeta^\pm M^{-1/2}$ and $|s_\pm| = \sup_{v_y \lesseqgtr 0, z} |s_\pm(v)|$. By (5.25), using the Schwartz inequality and

$$\int_{-1}^{1} dz \frac{1}{\varepsilon} \frac{1}{v_3(v,1,z)} H(z, v(1,z)) \exp -\frac{1}{\varepsilon}\Omega_{1,z}(v) < C \int_0^\infty dz \exp[-z] < C,$$

we get

$$\begin{aligned} |\int_{v_z>0} v_z M^{1/2} TH\big| & \leq \varepsilon^{-1/2} C \int_{v_z>0} v_z M^{1/2} \Big[\int_{-1}^{1} dy \nu^{-1} \frac{1}{v_3(v,1,z,)} H^2\Big]^{1/2} \\ & \leq \varepsilon^{-1/2} C \int_{v_z>0} |v_z|^{1/2} M^{1/2} \Big[\int_{-1}^{1} dy \nu^{-1} H^2\Big]^{1/2} \end{aligned}$$

Finally, using again the Schwartz inequality and recalling the expression of β_R in (5.14) we get

$$\beta_R \leq c[\varepsilon^{-1/2} \| \nu^{-1} H \|_2 + |s|]. \tag{5.33}$$

Using now the form of H in (5.28)

$$\beta_R \leq c[\varepsilon^{-1/2} \parallel \nu^{-1}\Phi \parallel_2 + \parallel \varepsilon D \parallel_2 + |s|].$$

As a consequence (5.31) becomes

$$|R\tilde{M}^{-1/2}|_r \leq C\sqrt{\varepsilon}|\tilde{M}^{-1/2}D|_{r-1} + C|\tilde{M}^{-1/2}\zeta^+|_r + |\tilde{M}^{-1/2}\zeta^-|_r \tag{5.34}$$

which concludes the analysis of the linear case. The non-linear problem is dealt with by a fixed point argument and the final result is

Theorem 5.5

There exist $\lambda > 0$, $\varepsilon_0 > 0$ and a constant C such that, if $\lambda < \lambda_0$ and $\varepsilon < \varepsilon_0$, there is a solution to the boundary value problem (5.4), (3.20), (3.21), (5.6) such that for any $r \geq 0$

$$|M^{-1/2}R|_r \leq C\varepsilon^{3/2}|M^{-1/2}A|_r \tag{5.35}$$

This concludes the proof of Theorem 5.1.

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Appendix

In this appendix we show how to get L_2 estimates for the boundary terms. We will prove (4.24). To this end it is enough to prove

$$S^+ := \varepsilon \int_0^{t_1} dt \int_{\mathbb{T}^2} dxdy \int_{v_z>0} dvv_z|\Phi|^2(x,y,1,v;t) \le \varepsilon^4 \int_0^{t_1} dt \parallel \frac{H_t}{\sqrt{\nu}} \parallel^2 + \sup_{0<t\le t_1} \zeta(\,\cdot\,,t)|^2 \tag{A.1}$$

where $H_t(\,\cdot\,) := H(\,\cdot\,,t)$ is defined in (4.44). In fact by substituting the expression of H_t we get

$$\begin{aligned}\varepsilon^4 \parallel \nu^{-1/2} H_t \parallel^2 \le &\parallel \nu^{-1/2}\tilde{K}\Phi_t \parallel^2 + \varepsilon^2 \parallel \nu^{-1/2}\tilde{K}^1 \Phi_t \parallel^2 \\ &+ \varepsilon^4 \parallel \nu^{-1/2}\tilde{K}^2\Phi_t \parallel^2 + \varepsilon^4 \parallel \tilde{D}_t \parallel^2 \le C \parallel \Phi_t \parallel^2 + \varepsilon^4 \parallel \tilde{D}_t \parallel^2\end{aligned}$$

so proving (4.24).

We use (4.37) to express the value of the function Φ in the point $z = 1$. We remark that the characteristic φ which is on the boundary $z = 1$ with velocity $v, v_z > 0$ at time $t^- - t$ had to start at time 0 either from some point in the bulk or from the boundary $z = -1$. In the first case the boundary term h in (4.37) is zero otherwise we get $\Phi(x', y', -1, v'; t)$, $v'_z > 0$ where

$$x' = x + v_x(t^- - t), \quad y' = x + v_y(t^- - t), \quad v' = v - G(t^- - t).$$

Hence

$$\begin{aligned}|\Phi|^2(x,y,1,v;t) \le 2|\Phi|^2\big(t^-, x', y', -1, v'\big)\big) \exp\Big\{ -2\int_{t^-}^t ds \frac{1}{\varepsilon^2}\nu\big(\varphi_{s-t}(x,y,1,v)\big)\Big\} \\ + 2\Bigg[\int_{t^-}^t dsH\big(s, \varphi_{s-t}(x,y,1,v)\big) \exp\Big\{ -\int_s^t ds' \frac{1}{\varepsilon^2}\nu\big(\varphi_{s'-t}(x,y,1,v)\big)\Big\}\Bigg]^2.\end{aligned} \tag{A.2}$$

By the boundary condition on Φ we have

$$\Phi\big(t^-, x', y', -1, v'\big)\big) = \alpha_R^-(t^-, x', y', -1)\overline{M}_-(v')\tilde{M}^{-1/2}(v') + \zeta^-(v'), \quad v'_z > 0$$

so that by Schwartz inequality we get

$$\int_{v'_z>0} v'_z dv' |\Phi|^2\big(t^-, x', y', -1, v'\big)\big) \le C(\alpha_R^-)^2(t^-, x', y', -1) + C\int_{v_z>0} v_z dv |\zeta^-|^2$$

But α_R^- is expressed again in term of the value of Φ on the boundary $z = -1$ and for negative z-component of the velocity, namely

$$(\alpha_R^-)^2 = \left[\int_{v_z<0} |v_z| \sqrt{\tilde{M}} \Phi(t^-, x', y', -1, v) \right]^2 \leq C \int_{v_z<0} |v_z| \, |\Phi|^2(t^-, x', y', -1, v)$$

We observe now that the integral

$$\int_{t^-}^{t} ds \nu\big(\varphi_{s-t}(x, y, -1, v)\big)$$

when evaluated on the characteristics going from one boundary to another is bounded from below. In fact we can use the bound on ν , $\nu(x, v) \geq c_0(1 + |v|)$ to check that this integral for the trajectory perpendicular to the boundary, namely $v_x = v_y = 0, v_z > 0$ is greater than ε times the width of the slab and this is the worst case since for the other trajectories is even greater. As a consequence, we can estimate the exponential in the first row of (A.2) as $e^{-C/\varepsilon}$. In conclusion we get

$$\begin{aligned} &\varepsilon \int_0^{t_1} dt \int dxdy \int_{v_z<0} dv |v_z| |\Phi|^2 \big(t^-, x', y', -1, v')\big) \exp\Big\{ -2 \int_{t^-}^{t} ds \frac{1}{\varepsilon^2} \nu\big(\varphi_{s-t}(x, y, 1, v)\big)\Big\} \\ &\leq C e^{-C/\varepsilon} \left[\int_0^{t_1} dt \int dxdy \int_{v_z<0} dv |v_z| \, |\Phi|^2 \big(t, x, y, -1, v)\big) + \sup_{0 \leq t \leq t_1} \parallel \zeta^- \parallel^2 \right]. \end{aligned} \tag{A.3}$$

To get the previous expression we have replaced x', y' and t^- with x, y and t, since the respective Jacobians are equal to one.

In this way we still do not have any explicit estimate of the boundary term in (A.2), because it contains the values of Φ at $z = -1$ for negative v_z which is still unknown. However, we can use the representation (4.37) again to express it back in terms of the function in $z = 1$ so to get a set of coupled equations for the boundary terms. Hence we consider

$$S^- := -\varepsilon \int_0^{t_1} dt \int_{\mathbb{T}^2} dxdy \int_{v_z<0} dv v_z |\Phi|^2(x, y, -1, v; t) \tag{A.4}$$

Equation (A.3) implies

$$S^+ \leq C e^{-C/\varepsilon} [S^- + \sup_{0 \leq t \leq t_1} \parallel \zeta^- \parallel^2] + \varepsilon \mathcal{H}^+ \tag{A.5}$$

where

$$\mathcal{H}^{\pm} = \int_0^{t_1} dt \int_{\mathbb{T}^2} dxdy \int dv|v_z| \Big[\int_{t^-}^{t} dsH\big(s, \varphi_{s-t}(x,y,\pm 1, v)\big)$$

$$\exp\Big\{ - \int_s^t ds' \frac{1}{\varepsilon^2} \nu\big(\varphi_{s'-t}(x,y,\pm 1,v)\big)\Big\}\Big]^2$$

As above we use (4.37) to find a bound for S^-. Define

$$E = \frac{|v|^2}{2\varepsilon^2} + \frac{1}{\varepsilon} G(z+1).$$

We have:

for $v_z < 0$, $E > 0$

$$\Phi(x,y,-1,v;t) = \Phi\big(t^-, x', y', 1, v'\big)\big) \exp\Big\{ - \int_{t^-}^{t} ds \frac{1}{\varepsilon^2} \nu\big(\varphi_{s-t}(x,y,-1,v)\big)\Big\}$$
$$+ \int_{t^-}^{t} dsH\big(s, \varphi_{s-t}(x,y,-1,v)\big) \exp\Big\{ - \int_s^t ds' \frac{1}{\varepsilon^2}\nu\big(\varphi_{s'-t}(x,y,-1,v)\big)\Big\}. \tag{A.6}$$

For $v_z < 0$, $E \leq 0$

$$\Phi(x,y,-1,v;t) = \Phi\big(t^-, x", y", -1, v"\big)\big) \exp\Big\{ - \int_{t^-}^{t} ds \frac{1}{\varepsilon^2} \nu\big(\varphi'_{s-t}(x,y,-1,v)\big)\Big\}$$
$$+ \int_{t^-}^{t} dsH\big(s, \varphi'_{s-t}(x,y,-1,v)\big) \exp\Big\{ - \int_s^t ds' \frac{1}{\varepsilon^2}\nu\big(\varphi'_{s'-t}(x,y,-1,v)\big)\Big\}. \tag{A.7}$$

where $(x'', y'', -1, v'') = \varphi_{t^- - t}(x,y,-1,v)$ and φ' is the characteristics starting from -1 with $v''_z > 0$.

These two formulas correspond to the case in which the characteristics starts from $z = 1$ with velocity $v'_z < 0$ and to the case in which it starts from $z = -1$ with $v''_z > 0$ and then come back to the boundary $z = -1$. This second possibility appears because of the presence of the force: the kinetic energy is such that the trajectory does not reach the opposite boundary but instead goes back after a time depending on the balance between kinetic energy and potential energy. In that case the total length ℓ for a trajectory with $v_x = 0 = v_y$ is given by

$$\ell = \frac{1}{2\varepsilon G}[v''_z]^2.$$

The other trajectories give a larger value to this integral.

Hence the square of the second term in the first row of (A.7), integrated on time, space and velocity, is bounded as

$$
\begin{aligned}
B :=& \int_0^{t_1} dt \int_{\mathbb{T}^2} dxdy \int_{v_z<0} |v_z''|\Phi|^2(x'',y'',-1,v'';t)\exp\{-\frac{C}{\varepsilon^2}v_z^2\} \\
=& \int_0^{t_1} dt \int_{\mathbb{T}^2} dxdy \int_{v_z>0} |v_z|\Phi|^2(x'',y'',-1,v_x'',v_y'',v_z;t)\exp\{-\frac{C}{\varepsilon^2}v_z^2\},
\end{aligned} \tag{A.8}
$$

using that $v_z'' = -v_z$. The function Φ in the integral in (A.8) is relative to $z=-1$ and $v_z>0$ so that we can express it in terms of α_R^- as follows, for $v_z>0$:

$$
\begin{aligned}
|\Phi|^2(x'',y'',-1,&v_x'',v_y'',v_z;t) \le {\alpha_R^-}^2 \bar{M}_-^2\tilde{M}^{-1} + (\zeta^-)^2, \\
&\le C\left[\int_{v_z<0} dv|v_z|\sqrt{\tilde{M}}\Phi(x'',y'',-1,v_x'',v_y'',v_z;t)\right]^2 \tilde{M} + {\zeta^-}^2 \\
&\le C\tilde{M}\int_{v_z<0} dv|v_z||\Phi|^2(x'',y'',-1,v_x'',v_y'',v_z;t) + (\zeta^-)^2
\end{aligned} \tag{A.9}
$$

Equation (A.9) implies for B the following bound

$$
\begin{aligned}
B \le& C\int_0^{t_1} dt \int dxdy \int_{v_z<0} dv|v_z||\Phi|^2(x'',y'',-1,v_x'',v_y'',v_z;t) \\
&\int dv|v_z|M\exp\{-\frac{C}{\varepsilon^2}v_z^2\} + C\int dv|v_z \parallel \zeta^-|^2\exp\{-\frac{C}{\varepsilon^2}v_z^2\}
\end{aligned} \tag{A.10}
$$

We have that

$$
\int dv|v_z|\tilde{M}\exp\{-\frac{C}{\varepsilon^2}v_z^2\} \le C\varepsilon^2 \tag{A.11}
$$

Finally

$$
\begin{aligned}
B \le& \varepsilon^2 C\int_0^{t_1} dt \int dxdy \int_{v_z<0} dv|v_z||\Phi|^2(x'',y'',-1,v_x'',v_y'',v_z;t) + \varepsilon^2 \sup_{0\le t\le t_1} \parallel \zeta^- \parallel^2 \\
=& C\varepsilon S^- + \varepsilon^2 \sup_t \parallel \zeta^- \parallel^2
\end{aligned}
$$

The contribute of the first term in (A.7) to S^- is

$$\varepsilon\int_0^{t_1}dt\int dxdy\int_{v_z<0}dv|v_z||\Phi|^2(x',y',1,v';t^-)\exp\Big\{-\int_{t^-}^t ds\frac{1}{\varepsilon^2}\nu\big(\varphi_{s-t}(x,y,-1,v)\big)\Big\}\le$$
$$\varepsilon\int_0^{t_1}dt\int dxdy\int_{v_z<0}dv|v_z|\big[\tilde{M}|\alpha_R^+|^2(x',y',1;t^-)+|\zeta^+|^2\big]$$
$$\times\exp\Big\{-\int_{t^-}^t ds\frac{\nu}{\varepsilon^2}\big(\varphi_{s-t}(x,y,-1,v)\big)\Big\}\le$$
$$Ce^{-C/\varepsilon}\left[\int_0^{t_1}dt\int dxdy\int_{v_z>0}dv|v_z||\Phi|^2(x',y',1,v';t)+\sup_{0\le t\le t_1}\parallel\zeta^+\parallel^2\right]$$
$$=Ce^{-C/\varepsilon}[S^++\varepsilon\sup_{0\le t\le t_1}\parallel\zeta^+\parallel^2] \tag{A.12}$$

Summarizing we can write

$$S^-\le C\varepsilon[S^-+\exp[-C/\varepsilon]S^++\varepsilon\sup_{0\le t\le t_1}\parallel\zeta^-\parallel^2]+\varepsilon\mathcal{H} \tag{A.13}$$

where $\mathcal{H}=\mathcal{H}^++\mathcal{H}^-$. This implies, for ε small,

$$S^-\le C\varepsilon^{-C/\varepsilon}[S^++\varepsilon\sup_{0\le t\le t_1}\parallel\zeta\parallel^2]+\varepsilon\mathcal{H} \tag{A.14}$$

Putting together (A.6) and (A.13) we get

$$\begin{aligned}S^+&\le C\left(\varepsilon\mathcal{H}+\sup_{0\le t\le t_1}\parallel\zeta\parallel^2\right),\\ S^-&\le C\left(\varepsilon\mathcal{H}+\sup_{0\le t\le t_1}\parallel\zeta\parallel^2\right).\end{aligned} \tag{A.15}$$

To conclude the argument we now provide an estimate for $\mathcal{H}$. By the Schwartz inequality for the integral on ds, we have

$$\begin{aligned}|\mathcal{H}^\pm|\le&\int_0^{t_1}dt\int dxdy\int dv|v_z|\int_{t^-}^t ds\Big|\frac{H}{\sqrt{\nu}}\Big|^2\big(s,\varphi_{s-t}(x,y,\pm1,v)\big)\\ &\int_{t^-}^t ds\nu\big(s,\varphi_{s-t}(x,y,\pm1,v)\big)\exp\Big\{-\int_s^t ds'\frac{2}{\varepsilon^2}\nu\big(\varphi_{s'-t}(x,y,\pm1,v)\big)\Big\}\le\\ &\le C\varepsilon^2\int_0^{t_1}dt\int dxdy\int dv|v_z|\int_{t^-}^t ds\Big|\frac{H}{\sqrt{\nu}}\Big|^2\big(s,\varphi_{s-t}(x,y,\pm1,v)\big)\end{aligned} \tag{A.16}$$

Consider the term

$$\mathcal{V}^{\pm} := \int_0^{t_1} dt \int_{t^-}^{t} ds \int dv |v_z| \Big|\frac{H}{\sqrt{\nu}}\Big|^2 \big(s, \varphi_{s-t}(x, y, \pm 1, v)\big)$$

and the following change of variables $(t, v_z) \to (\xi, w)$

$$\xi = \xi(t) := \pm 1 + \frac{1}{\varepsilon} v_z (t - s) + \frac{1}{2\varepsilon} G(t - s)^2, \qquad w = v_z + G(t - s)$$

whose Jacobian is $\varepsilon |v_z|^{-1}$. Denote also by $t(\xi)$ the inverse of $\xi(t)$. We have

$$\mathcal{V}^{\pm} := \varepsilon \int_{\xi(0)}^{\xi(t_1)} d\xi \int_{t^-(\xi)}^{t(\xi)} ds \int dw \Big|\frac{H}{\sqrt{\nu}}\Big|^2 \big(s, x(\xi, w), y(\xi, w), \xi, v_y, v_x, w)\big)$$

Hence

$$|\mathcal{H}| \le \varepsilon^3 \int_0^{t_1} ds \int_{\Omega} dx \int dv \Big|\frac{H}{\sqrt{\nu}}\Big|^2 \big(s, x, v\big). \tag{A.17}$$

which implies the bound (A.1).

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